



A MIXED-INTEGER LINEAR PROGRAMMING APPROACH TO THE
AC OPTIMAL POWER FLOW IN DISTRIBUTION SYSTEMS

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Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Engenharia Elétrica, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Engenharia Elétrica.

Orientador: Carmen Lucia Tancredo Borges

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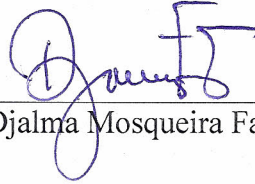
Rafael de Sá Ferreira

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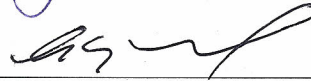
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UMA ABORDAGEM DE PROGRAMAÇÃO INTEIRA MISTA PARA O FLUXO DE POTÊNCIA ÓTIMO CA EM REDES DE DISTRIBUIÇÃO

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Orientador: Carmen Lucia Tancredo Borges

Programa: Engenharia Elétrica

O problema de fluxo de potência ótimo em redes de corrente alternada (FPO-CA) está dentre as ferramentas computacionais necessárias para o suporte à tomada de decisão no contexto do planejamento da operação e expansão de sistemas de distribuição. Nesta dissertação, emprega-se técnicas de linearização e convexificação para obter uma reformulação da versão não-linear do FPO-CA como um problema de programação inteira linear mista (PLIM). A formulação proposta: (i) captura o comportamento não-linear do sistema de distribuição através de aproximação cuja acurácia pode ser arbitrada pelo usuário; (ii) dá suporte a decisões discretas e contínuas; (iii) é construída com base em variáveis convencionalmente utilizadas para a descrição do comportamento da rede elétrica, o que resulta em flexibilidade na definição de funções objetivo e estende a aplicabilidade da formulação proposta a um conjunto elevado de problemas; e (iv) pode ser tratada por meio de pacotes comerciais para a solução de problemas de programação inteira mista, podendo-se obter soluções ótimas globais. Características físicas específicas de sistemas de distribuição são extensamente exploradas para obter-se uma formulação PLIM que concilie acurácia e desempenho computacional. A aplicabilidade e as características principais da formulação proposta são demonstradas com o auxílio de estudos de caso.

Abstract of Dissertation presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Master of Science (M.Sc.).

A MIXED-INTEGER LINEAR PROGRAMMING APPROACH TO THE
AC OPTIMAL POWER FLOW IN DISTRIBUTION SYSTEMS

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The alternating current (AC) optimal power flow (ACOPF) is among the computational tools required to support decision making in distribution system operations and expansion planning. In this dissertation, linearization and convexification techniques are employed in order to reformulate the non-linear version of the ACOPF for distribution systems, and a mixed-integer linear programming reformulation of this problem is proposed. The proposed formulation: (i) captures the non-linear behavior of the distribution system with an arbitrarily accurate approximation, with attention to the AC nature of the distribution system; (ii) supports both continuous and discrete decisions; (iii) is constructed with basis on conventional physical variables that describe network behavior, yielding significant flexibility in the definition of objective functions and extending its applicability to a number of different problems; and (iv) can be solved to global optimality with the use of widely employed and commercially available mixed-integer linear optimization solvers. Specific physical characteristics of distribution systems are extensively explored for achieving a MILP formulation that conciliates the desired attributes of accuracy and computational performance. The applicability and the main characteristics of the proposed formulation are showcased with help of several case studies.

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NOMENCLATURE

The nomenclature presented in the following is used in chapters 2 and 4.

Indices and sets

$k ; m$	Indices for buses of the distribution system.
km	Index for branches of the distribution system (this is the concise notation for the ordered pair $\langle k, m \rangle$, in which the first entry corresponds to the <i>from</i> bus of a given branch, and the second entry corresponds to the <i>to</i> bus of a the same branch).
$r ; s$	Index for <i>evaluation points</i> and associated variables, used in a number of different piecewise-linear approximations.
Γ^{re}	Set of indices for <i>evaluation points</i> $\hat{V}_k^{re,r}$ and associated variables.
Γ^{im}	Set of indices for <i>evaluation points</i> $\hat{V}_k^{im,s}$ and associated variables.
E^V	Set of indices for <i>evaluation points</i> \hat{V}_k^r and associated variables.
E^{lg}	Set of indices for <i>evaluation points</i> $\hat{I}_{g,k}^s$ and associated variables.
Π^{re}	Set of indices for <i>evaluation points</i> $\hat{I}_{km}^{re,r}$ and associated variables.
Π^{im}	Set of indices for <i>evaluation points</i> $\hat{I}_{km}^{im,s}$ and associated variables.
Ψ_C	Set of all branches in the distribution system.
Ψ_{CD}	Set of circuits that represent candidate reinforcements (candidate current-carrying facilities).
Ψ_{SW}	Set of switchable branches in the system.
Ω_B	Set of all buses in the distribution system.
Ω_{CAP}	Set of buses with candidate capacitors.
Ω_{CTRPQ}	Set of buses to which generators with control over the output of active and reactive power connect.
Ω_{CTRQ}	Set of buses to which generators with control only over reactive power output connect.
Ω_{CURT}	Set of buses to which curtailable generators connect.
Ω_{GEN}	Set of all buses to which generators (of any type) connect.

Ω_{ICTE}	Set of buses to which loads of the <i>constant-current</i> type connect.
Ω_{ITFC}	Set of buses at the interface of the internal network with the external network
Ω_{LOAD}	Set of all buses to which loads (of any type) connect.
Ω_k	Set of buses directly connected to bus k .
Ω_{NSHD}	Set of all buses to which loads that cannot be shed connect.
Ω_{PCTE}	Set of buses to which loads of the <i>constant-power</i> type connect.
Ω_{REF}	Set of voltage reference buses in the system.
Ω_{ROOT}	Set of buses chosen as <i>root buses</i> .
Ω_{SLACK}	Set of all slack buses in the system.
Ω_{SHED}	Set of all buses to which loads that can be shed connect.
Ω_{ZCTE}	Set of buses to which loads of the <i>constant-impedance</i> type connect.

Parameters

C_k^{CAPL}	Cost coefficient associated with the placement of the candidate capacitor at bus k (in \$).
C_{km}^{CONST}	Cost associated with construction of reinforcement represented by circuit km (in \$).
C_k^{CURT}	Cost coefficient associated with curtailment of generator at bus k (in \$/p.u.).
C_k^{GEN}	Cost coefficient associated with generation with controllable active power output at bus k (in \$/p.u.).
C_k^{IMPORT}	Cost coefficient associated with imports from the external network, at the interface represented as the slack bus k (in \$/p.u.).
C^{LOSS}	Cost coefficient associated with ohmic losses (in \$/p.u.).
C_k^{SHED}	Cost coefficient associated with load shedding at bus k (in \$/p.u.).
C_{km}^{SWITCH}	Cost of switching action (cost of changing the status of the switchable circuit) associated with circuit km (in \$).
d_k^P	Nominal value of active power demanded by load at bus k (in p.u.).
d_k^Q	Nominal value of reactive power demanded by load at bus k (in p.u.).

g_k^P	Active power generated by generator at bus k (in p.u.). This is a parameter for all generators in set Ω_{CTRPQ} , and a decision variable for all generators in set Ω_{CTRPQ} .
$\underline{g}_k^P ; \overline{g}_k^P$	Lower and upper bounds for active power output of generator at bus k (in p.u.). Defined only for generators in Ω_{CTRPQ} .
$\underline{g}_k^Q ; \overline{g}_k^Q$	Lower and upper bounds for reactive power output of generator at bus k (in p.u.).
$\hat{I}_{g,k}^s$	Evaluation points of real component of slack current of bus k in Ω_{ITFC} .
$\underline{I}_{g,k}^{re} ; \overline{I}_{g,k}^{re}$	Lower and upper bounds for the real component of the <i>slack current</i> of bus k in Ω_{ITFC} (in p.u.).
$\underline{I}_{km} ; \overline{I}_{km}$	Lower and upper bounds for magnitude of current flowing through branch km (in p.u.).
$M_k^{G,re,1} ; M_k^{G,re,2} ; M_k^{G,re,3} ; M_k^{G,re,4} ; M_k^{G,im,1} ; M_k^{G,im,2} ; M_k^{G,im,3} ; M_k^{G,im,4}$	Disjunctive constants for the disjunctive constraints employed for modeling generation curtailment.
$M_k^{D,re,ICTE,1} ; M_k^{D,re,ICTE,2} ; M_k^{D,re,ICTE,3} ; M_k^{D,re,ICTE,4} ; M_k^{D,im,ICTE,1} ; M_k^{D,im,ICTE,2} ; M_k^{D,im,ICTE,3} ; M_k^{D,im,ICTE,4}$	Disjunctive constants for the disjunctive constraints employed for modeling shedding of loads of the constant-current type.
$M_k^{D,re,PCTE,1} ; M_k^{D,re,PCTE,2} ; M_k^{D,re,PCTE,3} ; M_k^{D,re,PCTE,4} ; M_k^{D,im,PCTE,1} ; M_k^{D,im,PCTE,2} ; M_k^{D,im,PCTE,3} ; M_k^{D,im,PCTE,4}$	Disjunctive constants for the disjunctive constraints employed for modeling shedding of loads of the constant-power type.
$M_k^{D,re,ZCTE,1} ; M_k^{D,re,ZCTE,2} ; M_k^{D,re,ZCTE,3} ; M_k^{D,re,ZCTE,4} ; M_k^{D,im,ZCTE,1} ; M_k^{D,im,ZCTE,2} ; M_k^{D,im,ZCTE,3} ; M_k^{D,im,ZCTE,4}$	Disjunctive constants for the disjunctive constraints employed for modeling shedding of loads of the constant-impedance type
$M_k^{IOFl1} ; M_k^{IOFu1} ; M_k^{IOFl2} ; M_k^{IOFu2}$	Disjunctive constants for disjunctive constraints for product $V_k \cdot (1 - \rho_k)$.

$M_k^{ZOF11} ; M_k^{ZOFu1} ; M_k^{ZOF12} ; M_k^{ZOFu2}$	Disjunctive constants for disjunctive constraints for product $\mu_k \cdot (1 - \rho_k)$.
$\hat{P}_k^{r,s}$	Evaluated values of function $P_k(V_k, I_{g,k}^{re})$, for bus k .
R_{km}	Resistance of the branch connecting buses k and m (in p.u.).
R_k^l	Resistance of constant impedance load at bus k (in p.u.).
$\hat{V}_k^{re,r}$	Evaluation point of real component of voltage at bus k (in p.u.).
$\hat{V}_k^{im,s}$	Evaluation point of imaginary component of voltage at bus k (in p.u.).
$\hat{V}_k^{r,s}$	Evaluated values of function $V_k(V_k^{re}, V_k^{im})$, for bus k (in p.u.).
\hat{V}_k^r	Evaluation points of voltage magnitude of bus k in Ω_{ITFC} .
$\underline{V}_k ; \overline{V}_k$	Lower and upper bound for magnitude of voltage at bus k (in p.u.).
V_k^{ref}	Fixed voltage magnitude of reference bus k (in p.u.).
W_{km}^{CL}	Disjunctive constant for Kirchhoff's Current Law.
$W_{km}^{VL,re,l} ; W_{km}^{VL,re,u} ; W_{km}^{VL,im,l} ; W_{km}^{VL,u}$	Disjunctive constants for Kirchhoff's Voltage Law.
X_{km}	Reactance of the branch connecting buses k and m (in p.u.).
X_k^l	Reactance of constant impedance load at bus k (in p.u.).
Z_k^l	Impedance of constant impedance load at bus k (in p.u.).
θ_k^{ref}	Reference angle for reference bus voltage at bus k (in degrees).
$\hat{l}_{km}^{re,r}$	Evaluation points of l_{km}^{re} , for branch km (in p.u.).
$\hat{l}_{km}^{im,s}$	Evaluation points of l_{km}^{im} , for branch km (in p.u.).
$\hat{l}_{km}^{r,s}$	Evaluated values of function l_{km} for branch km (in p.u.).
$\hat{\mu}_k^{r,s}$	Evaluated values of function V_k^2 (in p.u. ²).
$\hat{\eta}_k^{r,s}$	Evaluated value of function $\eta_k(V_k^{re}, V_k^{im})$, for bus k (dimensionless).
$\hat{\kappa}_k^{r,s}$	Evaluated value of function $\kappa_k(V_k^{re}, V_k^{im})$, for bus k (dimensionless).
$\hat{\xi}_k^{r,s}$	Evaluated values of function $\xi_k(V_k^{re}, V_k^{im})$, for bus k (in 1/p.u.).
$\underline{\xi}_k ; \overline{\xi}_k$	Lower and upper bounds for the values that ξ_k may assume.
$\hat{\zeta}_k^{r,s}$	Evaluated values of function $\zeta_k(V_k^{re}, V_k^{im})$, for bus k (in 1/p.u.).
$\underline{\zeta}_k ; \overline{\zeta}_k$	Lower and upper bounds for the values that ζ_k may assume.

Continuous decision variables

g_k^P	Active power generated by generator at bus k (in p.u.), free in signal or non-negative depending on the upper and lower bounds defined. This is a decision variable for all generators in Ω_{CTRPQ} .
g_k^Q	Reactive power generated by generator at bus k (in p.u.), free in signal.
$I_{d,k}^{re}$	Real component of current demanded by load connected to bus k (in p.u.), free in signal.
$I_{d,k}^{im}$	Imaginary component of current demanded by load connected to bus k (in p.u.), free in signal.
$I_{g,k}^{re}$	Real component of current generated by generator connected to bus k (in p.u.), free in signal.
$I_{g,k}^{im}$	Imaginary component of current generated by generator connected to bus k (in p.u.), free in signal.
I_{km}	Magnitude of current flowing through branch km (in p.u.), non-negative.
I_{km}^{re}	Real component of current flowing through the branch connecting buses k and m , from bus k to bus m (in p.u.), free in signal.
I_{km}^{im}	Imaginary component of current flowing through the branch connecting buses k and m , from bus k to bus m (in p.u.), free in signal.
$I_{d,k}^{im}$	Imaginary component of current demanded by load connected to bus k (in p.u.), free in signal.
$m_k^{P,re}$	Auxiliary decision variable for modeling the product $\xi_k \cdot g_k^P$ (in p.u.), free in signal or non-negative depending on the upper and lower bounds defined for g_k^P .
$m_k^{P,im}$	Auxiliary decision variable for modeling the product $\zeta_k \cdot g_k^P$ (in p.u.), free in signal.
$m_k^{Q,re}$	Auxiliary decision variable for modeling the product $\zeta_k \cdot g_k^Q$ (in p.u.), free in signal.
$m_k^{Q,im}$	Auxiliary decision variable for modeling the product $\xi_k \cdot g_k^Q$ (in p.u.), free in signal.

P_k	Auxiliary (continuous) decision variable for approximating the product $V_k \cdot I_{g,k}^{re}$, for all buses k in Ω_{ITFC} (in p.u.), free in signal or non-negative depending on the upper and lower bounds defined.
V_k	Magnitude of voltage at bus k (in p.u.), non-negative.
V_k^{re}	Real component of voltage at bus k (in p.u.), non-negative.
V_k^{im}	Imaginary component of voltage at bus k (in p.u.), free in signal.
α_k	Continuous decision that assumes the value $\alpha_k = 1$ if and only if $\rho_k = 1$ and $\tau_k = 1$; and assumes the value $\alpha_k = 0$ for all other combinations of the binary variables ρ_k and τ_k . Dimensionless and non-negative.
$\varphi_k^{r,s}$	Weights for constructing piecewise-linear approximation of non-convex, non-linear function of V_k and $I_{g,k}^{re}$ (dimensionless), non-negative.
γ_k^{ICTE}	Auxiliary decision variable for modeling the product $V_k \cdot (1 - \rho_k)$ (in p.u.), non-negative.
γ_k^{ZCTE}	Auxiliary continuous decision variable for modeling the product $\mu_k \cdot (1 - \rho_k)$ (in p.u. ²), non-negative.
η_k	Auxiliary decision variable that models a function of the complex voltage components at bus k (dimensionless), non-negative.
l_{km}	Auxiliary variable that is at least as high as I_{km} , for branch km (in p.u.), non-negative.
l_{km}^{re}	Auxiliary variable that is at least as high as the modulus of I_{km}^{re} , for branch km (in p.u.), non-negative.
l_{km}^{im}	Auxiliary variable that is at least as high as the modulus of I_{km}^{im} , for branch km (in p.u.), non-negative.
κ_k	Auxiliary decision variable that models a function of the complex voltage components at bus k (dimensionless), free in signal.
$\lambda_k^{r,s}$	Weights for constructing piecewise-linear approximation of non-convex, non-linear functions of V_k^{re} and V_k^{im} (dimensionless), non-negative.
μ_k	Auxiliary variable that represents approximation of V_k^2 , for k in Ω_{ZCTE} (in p.u. ²), non-negative.
ξ_k	Auxiliary decision variable that models a function of the complex voltage components at bus k (in 1/p.u.), non-negative.

ζ_k	Auxiliary decision variable that models a function of the complex voltage components at bus k (in 1/p.u.), free in signal.
$\omega_{km}^{r,s}$	Weights for constructing piecewise-linear approximation of t_{km} , for branch km (dimensionless), non-negative.

Binary decision variables

$b_k^r ; c_k^s$	Auxiliary binary decision variables for ensuring that the weights $\varphi_k^{r,s}$ form a SOS2.
$x_k^r ; y_k^s$	Auxiliary binary decision variables for ensuring that the weights $\lambda_k^{r,s}$ form a SOS2.
$t_{km}^r ; u_{km}^s$	Auxiliary binary decision variables for ensuring that the weights $\omega_{km}^{r,s}$ form a SOS2.
β_k	Binary decision variable that models the decision to disconnect a bus k from the system (the generator is disconnected if $\beta_k = 0$), employed in connectivity approach (iii).
ρ_k	Binary variable that indicates if load at bus k is shed ($\rho_k = 1$ indicates that load is shed).
σ_{km}	Binary variable that represents the status of circuit km : if this is a candidate reinforcement, $\sigma_{km} = 1$ indicates that reinforcement is built; if this is a switchable branch, $\sigma_{km} = 1$ indicates that branch is switched-on.
τ_k	Binary variable that indicates if generator at bus k is curtailed ($\tau_k = 1$ indicates that generator is curtailed).
v_{km}^k	Binary variable associated to line km that assumes the value $v_{km}^k = 1$ if bus k is the <i>parent</i> of bus m , and that assumes the value $v_{km}^k = 0$ if bus m is the <i>parent</i> of bus k .

1 INTRODUCTION

This introductory chapter begins with an exposition of the background and the motivation for the development of the research that lead to this dissertation. In section 1.2, the technical literature on the research topic is reviewed. The objective and the technical contributions of this work are presented in section 1.3, and the chapter ends with a description of the organization of this document.

1.1 Background and motivation

In the course of the last decade, the evolution of business models have either brought about important challenges to the distribution segment of the electricity business, or enhanced the criticality of previously existing ones:

- (i) Performance-based regulation (or incentive-based regulation) [1]-[5] has been adopted in several jurisdictions with the objective of incentivizing operational efficiency and controlling costs perceived by electricity consumers. Utilities have thus received strong economic incentives to optimize the expansion of the distribution network and the use of existing distribution assets, which often resulted in pressure to operate the system closer to admissible technical limits.
- (ii) The development of information technology has fundamentally changed the requirements of retail consumers on the continuity and adequacy of electricity supply. This has been a motivation for the adoption of reliability-driven economic incentives for regulated distribution utilities in many jurisdictions around the globe, strictly binding compliance to technical performance standards (supply continuity and adequacy) [4]-[5] with the financial health of distribution utilities.

Also, technological advances associated with the evolution of the electrical system towards the smart grid have led to growing attention to the use of sensory information and automation within the distribution system. The deployment of these technologies is expected not only to facilitate the achievement of operational efficiency and adequate technical performance, but also to enhance the observability and controllability of the grid. This enhanced controllability is thought to be a feature that

will help distribution utilities to better integrate distributed generation and storage within their systems, and allow a more active participation of end-consumers (including those from the retail segment) in market and system operations [6]-[7].

In order to meet the challenges and achieve the goals listed in the previous paragraphs, distribution management systems require advanced computational tools [7]-[8] to support operation decisions with respect both to traditional processes (such as system reconfiguration or integrated voltage/reactive power control) and to new, envisioned functions (such as central control of distributed generation). But it is not only operations planning that drives the growing demand for advanced computational tools for distribution systems: the need for tools to support expansion planning decisions has also become more critical, due to the need to coordinate traditional activities, such as reinforcement to current-carrying facilities and placement of capacitors and switches, with the goals of asset optimization and accommodation of distributed generation, among others.

The alternating current (AC) Optimal Power Flow (ACOPF) is among the tools required for several of the distribution system operations and expansion processes listed in the previous paragraphs. In the ACOPF problem, one seeks the optimization of a given objective function (e.g. minimization of generation costs, minimization of costs of ohmic losses), subject to constraints that represent the physical laws governing power systems and the operating limits of network equipment. Explicit reference is made here to the AC nature of the problem, as a reminder that phenomena related to reactive power and to bus voltage magnitudes are of great relevance to distribution system expansion and operation [9].

The reader will notice that several of the distribution system expansion and operation processes listed in the previous paragraphs involve discrete decisions, such as circuit construction, placement of switches and system reconfiguration. It is obviously in the interest of distribution engineers that such discrete decisions are modeled within the ACOPF, in order to take full advantage of the optimization tools. However, due to the non-linear nature of the ACOPF, factoring discrete decisions into the optimization approach is a complex task.

As described in section 1.3, this dissertation aims at presenting a formulation for the ACOPF in distribution systems that is amenable to the incorporation of discrete

decisions, and that may thus be used to support a wide range of applications in operations and expansion processes.

1.2 Bibliographic review

This section provides the reader with a review of the technical literature on decision support tools for distribution system operations and expansion planning, with focus on the ACOPF problem and, particularly, on discrete decisions. As this dissertation is oriented towards the solution of the ACOPF in distribution systems, the bibliographic review will emphasize the *formulation and solution approach* employed in the references, regardless of whether each reference deals with single-stage or multi-stage applications, deterministic or stochastic problems, or other specific features that are more involved with the application than to the formulation and solution of the ACOPF problem.

Some of the earlier works on support systems involving discrete decisions for distribution planning and operation, such as [10], consist of computational tools that basically duplicate *ad hoc* heuristic analyses conducted by distribution system engineers. In reference [10], which is oriented towards distribution expansion planning, the proposed computational tool involves the sequential execution of procedures for: comparing load forecasts to substation capacity; determining preliminary (discrete) reinforcements for the relief of overloads, with help of heuristic procedures; checking the feasibility of the preliminary solutions with help of a *simplified load flow program* (the expression is used in [10] without further explanation); and finding solutions that remove technical infeasibilities, with help of further heuristics.

Solution algorithms based on heuristics have, indeed, been widely used in distribution system planning. One heuristic technique that has been widely used is the *branch exchange*. This technique, particularly employed in distribution system expansion and reconfiguration problems in which radiality constraints must be enforced, basically consists of starting from an arbitrary initial solution that complies with the radiality constraints, and then iteratively choosing a pair of branches to be *exchanged* – this meaning that, in each iteration, a branch that does not pertain to the active network topology is activated, and a branch that pertains to the active topology is deactivated. The pair of branches to be activated/deactivated is chosen with help of any given metric

that captures the sensitivity of the objective function (e.g., minimization of losses or of circuit overloading) with respect to changes in the status of branches. The method is usually based on local sensitivities, meaning that the calculation of the sensitivities is made considering the network topology verified at the beginning of each iteration. The iterative process is repeated until no branch exchanges that result in improvements to the objective function are found. Obviously, the *branch exchange* heuristic applies to problems in which the discrete decisions refer to modifications in circuit status (switched on/off for reconfiguration problems, and constructed/not constructed for expansion problems).

The authors of [11] make use of a technique such as the one described in the last paragraph (though not using the name *branch exchange*), in the context of feeder reconfiguration for loss reduction, and with local sensitivities basically obtained with help of the equations of the AC power flow problem, expressed in terms of complex branch currents and bus voltages. In [11], formulas for the estimating the local sensitivities with different levels of accuracy are presented.

Reference [12] presents a method for distribution system expansion planning that relies on local sensitivities – and the term *branch exchange* is actually used to describe the proposed method. The local sensitivities used for choosing the set of branches to be exchanged are calculated with help of linear programming techniques. In order to allow the use of linear programming techniques, the authors employ a “*direct current power flow calculation*” [12] to model distribution network behavior. It is worth mentioning that the authors initially present a mixed-integer linear programming formulation of the OPF in distribution systems, based on the *direct current power flow* formulation, before characterizing this approach as being excessively demanding to solve directly, and describing and employing the *branch exchange* method. The authors of [12] also employ the *branch exchange* method in [13], considering in the latter reference a multi-stage problem.

Heuristic methods based on the calculation of local sensitivities are also applied to other problems within distribution system expansion and operations planning. As an example, a method based partially in local sensitivities (which are used within a hybrid algorithm that combines *tabu search* with features from other metaheuristics¹) is

¹ Metaheuristics will be reviewed further in this section.

employed in [14] for the capacitor placement and sizing problem, in which network behavior is modeled via non-linear equations. Heuristic methods based on local sensitivities have also been employed to the problems of coordinated electric vehicle charging [15] and distributed generation placement and sizing [16], [17].

Heuristic methods based on successively and iteratively performing *greedy* searches, and in each iteration making a decision that most improves the value of a given metric, have been proposed for distribution systems applications, particularly for the network reconfiguration problems. The *branch exchange* heuristic is obviously an example of such methods. Other examples are those presented in [18] and [19]: in both methods, the solution algorithm is based on initially considering all switches closed and executing the iterative algorithm, in each iteration opening the switch that results in the largest improvement of a given metric. The methods employed in [18] and [19] differ from those reviewed in the previous paragraphs in that the evaluation of the metric is not based on sensitivity analyses considering the topology *at the beginning* of each iteration as fixed, but rather on implicit investigation of the changes in the evaluation metric that would be obtained *after* a switching decision would be made. In reference [18], the evaluation metric is obtained in each iteration via the solution of a modified ACOPF, with the simplifying assumption that all loads are current sources, and modeling the closed switches as fully adjustable current sources. After the solution of this modified ACOPF, the switch that carries the lowest current is selected for opening. In the approach of reference [19], the chosen metric is the value of overall system losses *after* the switching decision, and the choice of the switch to be opened in each iteration is made via implicit evaluation of all possibilities via the standard Newton method with second derivatives.

There are also classes of greedy algorithms for distribution system reconfiguration that include features for partially mitigating the problems of a *purely* greedy search. In the algorithm proposed in reference [20], all switches are initially considered opened, and local sensitivities of the proposed objective function (incremental losses divided by incremental load served) with respect to branch switching are used to screen candidates and make the choice of a single switch to be closed in each iteration. The solution of the full set of AC power flow equations is made after each closing action in order to ensure feasibility, and the authors propose a backtracking feature, based on the construction of lists and on ranking, to mitigate the

problems of a purely greedy search. Reference [21] proposes a method that starts with all switches closed, and then proceeds to screening candidates and preliminarily determining the switch to be opened by evaluating AC power flow equations. A heuristic based on the branch exchange technique is used to partially mitigate the short-sightedness of a purely greedy search.

A number of methods based on representing a switch by a continuous function, rather than a discrete (on/off) model, have been proposed for the distribution system reconfiguration problem. Reference [22] proposes a method in which all switches are initially considered closed, and heuristics are employed to iteratively open switches until the network is radial. The first step in the heuristic procedure for choosing the switch to open is an ACOPF in which switches are represented by a linear variable that may assume any value within the interval $[0, 1]$. The objective function of this ACOPF accounts for power losses and *branch utilization costs*. The optimal value of the continuous decision variable that represents switches is used for raking candidates, and posterior heuristics involving evaluation of full power flow equations, now with discretely modeling (on/off) of the status of short-listed switches, are executed to support a final decision on the switch to open in each iteration. In [23], switches are represented via sigmoidal functions, and a non-linear ACOPF is solved in each iteration of a heuristic that starts with a meshed topology and successively open switches, in order to achieve a radial topology. The Lagrange multipliers associated with specific constraints of the ACOPF are used for the ranking of switches to be opened, in the first steps of the heuristic procedure. Sigmoidal functions have also been used to model discrete decisions regarding capacitor placement [24], within a heuristic approach which is similar to that described above. It is worth mentioning that reference [23] treats both network reconfiguration *and* capacitor placement decisions.

The attention of the reader is now directed back to methods built upon local sensitivities. Besides being used in iterative heuristics, local sensitivities have also been used in methods that utilize classical optimization techniques (mainly linear programming and mixed-integer linear programming) to solve formulations of the ACOPF that are characterized by a local linearization around a pre-defined operating point.

As an example, the authors of [25] present a set of linear equations to solve the steady-state power flow problem in distribution networks, and propose a linear

programming model for the problem of minimizing losses in a distribution system with distributed generation. In the proposed formulation, the complex nature of state variables is taken into account while defining equations for the First and Second Kirchhoff Laws in rectangular coordinates. As the formulation is expressed in terms of complex voltages and *current* flows and injections (as opposed to complex voltages and *power* flows and injections), the current demanded by constant-power loads would be described via non-linear equations. The authors thus employ a representation of constant-power loads based on local linearization of the relationship among load currents and bus voltages, with help of approximate multiplicative factors determined offline (i.e., previously to the solution of the steady state power flow or to the optimization problem, and not within the problem solution). The solution approach proposed in [25] does not explicitly deals with discrete decisions variables. Another example of a method based on local linear approximations used within a linear programming approach is [26]. Reference [26] presents an expansion planning model with approximate and simplified modeling of network behavior, in which the voltage drop across a given branch is approximated as a real quantity, given by the product of the branch apparent power flow (in MVA) by a constant calculated offline (i.e., not within the solution of the optimization model), as a function of an assumed (lagging) power factor, branch impedance and rated voltage.

Simplified models of network behavior based on local linear approximations are also used within MILP approaches to the distribution system expansion planning problem. A number of references employ restrictive approximations regarding the complex nature of bus voltages and branch currents, while proposing MILP formulations. In [27], a mixed-integer expansion planning model that encompasses both the primary and the secondary distribution grids is presented. The authors of [27] suggest that constraints on voltage drops along sets of branches are explicitly enforced only for identified critical routes (a critical route being a set of branches that connects the voltage source to a bus with potential violations of voltage limits). They also suggest approximating those voltage drops as the product of apparent power flows by a multiplicative constant calculated offline, with basis on branch parameters and bus voltages obtained from a load flows solved previously to the optimization algorithm.

MILP approaches that employ other classes of approximations have also been proposed for distribution system expansion planning problems. For instance, reference

[28] proposes one such model, focusing on switch placement with the objective of minimizing capital investment and operation costs, with particular emphasis on interruption costs. Due to the exclusive focus on continuity, only the First Kirchhoff Law is modeled, with the Second Kirchhoff Law (the voltages law) purposefully not being incorporated to the model.

References [29] and [30] also present mixed-integer models for distribution system expansion planning, with particular attention respectively to distributed generation and to the treatment of reliability. In both of these references, voltage drops across branches are approximated by the real product of branch currents and branch impedances (which is a restrictive approximation), and all loads are modeled as fixed current injections. Modeling loads as fixed current injections may be interpreted as a linearization around a pre-defined operation point, due to the fact that the actual currents injections corresponding to constant-power and constant impedance loads vary according to bus voltages.

It is worth pointing out that simplifications such as representing voltage drops across branches by the real product of branch currents and branch impedances were also employed in mixed-integer programming approaches to distribution system expansion planning dated from the early 1980's, such as [31]. Other MILP formulations for distribution system expansion planning problems proposed in the early 1980's employ other classes of approximations regarding the network model. For instance, reference [32] focused exclusively in connectivity and balance of power while representing network behavior, not accounting for Kirchhoff's Voltage Law. Other models, such as [33], placed emphasis on the solution of the distribution expansion planning problem using pre-calculated, aggregate cost functions – the power-loss envelope curves defined in [33] –, with little attention to the representation of network behavior.

A number of other mathematical programming approaches, besides linear and mixed-integer programming methods, have been applied to distribution system operations and planning problems. References that employ such approaches are reviewed in the following.

In reference [34], which deals with the problem of service restoration in unbalanced three-phase distribution systems, the non-linearities associated with the AC OPF model are accommodated within a mixed-integer non-linear programming formulation. The authors point out that the solver LINGO [35] (citation obtained from

[34]), which treats mixed-integer non-linear programs with a branch-and-bound algorithm in which each node of the branch-and-bound tree is evaluated via successive linear programming, has been used for the solution of the proposed formulation.

Reference [36] also presents a mixed-integer non-linear programming formulation of the distribution system reconfiguration problem. The proposed formulation includes binary decisions modeling the connection/disconnection of capacitors and generating units. The solution approach involves two-stage Benders decomposition, in which all discrete decisions are treated within the master problem (which has a quadratic objective function due to the modeling of losses, and includes some of the linear network constraints), whereas the slave problem ensures feasibility with respect to (non-linear) network behavior. The master and slave problems are coupled via linear Benders cuts.

Variable transformations are an important technique employed in references [37] and [38]. Reference [37] presents a mixed-integer quadratically constrained programming formulation for the problem of distribution system reconfiguration to minimize ohmic losses. The exact formulation of [37] is based on defining nonconventional transformed variables in order to model network behavior. Finally, reference [38] deals with the problem of distribution network reconfiguration to minimize ohmic losses, presenting an exact convex second-order cone programming formulation for this problem, as well as a MILP formulation with polyhedral approximation of the conic constraints (for which auxiliary nonconventional variables are defined).

Metaheuristics have also been widely employed as solution approaches to distribution system operation and planning problems in recent times. The flexibility of these approaches allows modeling the full set of non-linear equations for the ACOPF, within several classes of problems. The most common approaches used in recent times include the methods listed below, used at times in combinations with other heuristics:

- *Genetic/evolutionary algorithms*: examples of references that make use of this technique include [39]-[50].
- *Simulated annealing*: examples of references that make use of this technique include [51]-[54].
- *Tabu search*: examples of references that make use of this technique include [55]-[57].

It should be noted that, for the specific problem of network reconfiguration for achieving minimal losses in radial distribution systems, a brute-force algorithm based on exhaustive search have been proposed in [58]. The authors employ graph-theoretic techniques, based on semi-sparse transformations of a current sensitivity matrix, to increase the efficiency of the exhaustive search method.

In complement to the previously mentioned references, the reader may find extensive reviews of distribution system planning models in [59] and [60], including works that deal with discrete decisions, but that were not treated in this section due to the similarity with at least one of the listed references.

Having concluded the bibliographic review, the objective and the technical contributions of this dissertation are presented in the following section.

1.3 Objective and contributions of this dissertation

The objective of this dissertation is to develop a mixed-integer linear programming (MILP) reformulation of the AC optimal power flow (ACOPF) problem for distribution systems that:

- (i) captures the non-linear behavior of the distribution system with an arbitrarily accurate approximation;
- (ii) supports both continuous and discrete decisions, respectively via continuous and integer decision variables;
- (iii) is constructed with basis on conventional physical variables that describe network behavior (bus voltages, branch currents, bus power injections, etc.), yielding significant flexibility in defining a number of possible objective functions for the ACOPF, and extending its applicability to a number of different problems faced by distribution system engineers; and
- (iv) can be solved to global optimality with the use of widely employed and commercially available mixed-integer linear optimization solvers.

Furthermore, as most commercially available mixed-integer linear optimization solvers have options to provide the user with detailed execution reports, including information on the duality gap displayed on-screen during execution, the user is able to control the quality of the solutions obtained in the course of the solution of the MILP

problem, eventually interrupting the optimization algorithm and obtaining an intermediate solution, for which the value of the duality gap is known (i.e., the quality of the solution can be controlled), if desired.

Convexification and linearization techniques will be extensively used to develop the MILP reformulation of the ACOPF problem for distribution systems, and the particular physical characteristics of the distribution system will be explored while applying these techniques, with the goal of enhancing its computational efficiency.

The technical contributions of this dissertation relate not only to the novelty of the proposed MILP reformulation of the ACOPF, but also to the fact that it *simultaneously* accounts for all aspects listed at the beginning of this section. The reader will notice that none of the methods presented in the reviewed references simultaneously displays the set of attributes (i)-(iv) listed before. The following points are highlighted:

- Despite the fact that many of the methods based on heuristics of metaheuristics generally lead to high-quality sub-optimal solutions, none of them present inherent guarantees of convergence to the global optimal solutions.
- Many of the methods based on classical mathematical programming techniques, particularly those that employ linear programming or mixed-integer programming, are based on severe and restrictive approximations of the non-linear behavior of the distribution network.
- Some methods based on mathematical programming apply techniques that are not currently available in the most commonly used commercial-grade optimization solvers. The possibility to use commercial optimization solvers is important for industry applications, due to the guarantee of longevity, maintainability and prevention of obsolescence of the solver that underlies practical utility applications.

At this point, it is worth mentioning that, in the technical literature, reference has already been made to the application of the linearization and convexification techniques used in this dissertation to power system problems. As an example, the authors of [61], while discussing the appropriateness of MILP reformulation of non-linear problems, make explicit reference to “*network problems with nonlinearities occurring on the edges such as the design and management of energy networks design*”, though not

providing any formulation of a specific problem. In fact, MILP reformulations have been proposed for the problem of the ACOPF in transmission systems, employing exclusively equations that are functions of voltage and power quantities [62]. However, to the knowledge of the author of this dissertation, no formulation directed to distribution systems, that employs equations that are functions of voltages and currents to describe network behavior, and that take specific characteristics of the distribution network into account in order to achieve adequate trade-offs between accuracy and computational performance, have been proposed.

The formulation proposed in this dissertation applies both to radial and to meshed distribution systems (a feature that lacks in many of the approaches listed in the bibliographic review, notably among those based in greedy heuristics, such as the branch exchange technique). However, the application of the proposed approach is currently limited to either three-phase balanced distribution systems or to single-phase networks.

1.4 Organization of the dissertation

The remainder of this dissertation is organized as follows:

- In chapter 2, the non-linear version of the ACOPF problem in distribution systems is presented. This chapter will begin with a discussion on the particular characteristics of distribution networks that are relevant for the formulation and solution of the optimal power flow problem. Selected applications of the ACOPF in distribution system operations and expansion planning are also presented.
- Convexification and linearization techniques for the reformulation of non-linear, non-convex problems (such as the ACOPF in distribution systems with discrete decisions) as mixed-integer linear programs are presented in chapter 3.
- The proposed MILP reformulation of the ACOPF for distribution systems is presented in details in chapter 4.
- The proposed formulation is applied to several case studies in chapter 5. The analysis of results of these case studies allows showcasing the

applicability of the proposed formulation and discussing its features and characteristics.

- Conclusions and suggestions for future work are presented in chapter 6.
- References are listed at the end of this document.
- The input data for the case studies of chapter 5 is presented in Appendix A (chapter 7).
- An alternative MILP reformulation of the ACOPF in distribution systems is presented in Appendix B (chapter 8).
- An alternative method for formulating the constraints through which the current injections of generators are obtained is presented in Appendix C (chapter 9).

2 THE (NON-LINEAR) ACOPF IN DISTRIBUTION SYSTEM OPERATIONS AND EXPANSION PLANNING

This chapter begins with the presentation of characteristics of the distribution system that are relevant for the formulation of the ACOPF problem.

The formulation of the non-linear version of the ACOPF problem for distribution systems (not yet including the modeling of discrete decisions²) is then presented in section 2.2.

The chapter ends with a list of selected applications of the ACOPF to distribution system expansion and operations planning.

2.1 Relevant characteristics of distribution systems

For the purposes of this dissertation, the distribution system is defined as the set of current-carrying facilities at rated voltages inferior to 69 kV that either functions as an isolated system or originates at step-down substations at the interface with the subtransmission or transmission network. In this definition, the distribution system includes all electrical power sources, loads and associated control equipment connected to the buses at rated voltages inferior to 69 kV. This definition is clearly oriented towards the ACOPF problem and by no means aims at being exhaustive – this is illustrated by the very fact that the definition does not coincide with that used in PRODIST [63]³, the grid code for electrical power distribution in Brazil.

The following subsections review particular characteristics of the distribution system, which are relevant to the formulation of the ACOPF problem (and particularly to its MILP reformulation, as will be seen in chapter 4).

² The representation of discrete decisions will be dealt with in chapter 4.

³ In PRODIST [63], the distribution system is defined as the set of electrical facilities and equipment owned by a distribution utility and located in its concession area, and may include facilities at voltage levels equal to and above 69 kV. According to PRODIST, the set of facilities with voltages below 69 kV would be defined as the union of the *medium voltage* and the *low voltage* distribution (sub)systems.

2.1.1 Shunt susceptance of overhead distribution lines

The first relevant characteristic of distribution systems is that the shunt susceptance of overhead distribution lines is comparatively lower than that of overhead transmission lines. This relates mainly to rated voltage levels and to constructive characteristics of distribution lines.

In fact, it is usual to consider the shunt susceptance of overhead distribution lines may as negligible in power flow calculations. In this case, circuits are represented exclusively by their series resistance and reactance – this approximation is considered, e.g., in [19], [25], [34], [64]-[67], and will also be adopted in this dissertation.

2.1.2 Resistance-to-reactance ratio

Also due to the comparatively lower voltage levels and to constructive characteristics, the typical resistance-to-reactance (R/X) ratio of overhead distribution lines is comparatively higher than the typical ratio of transmission and subtransmission lines.

This has important implications for the power flow analyses in distribution systems, which will be discussed with help of Figure 2.1.

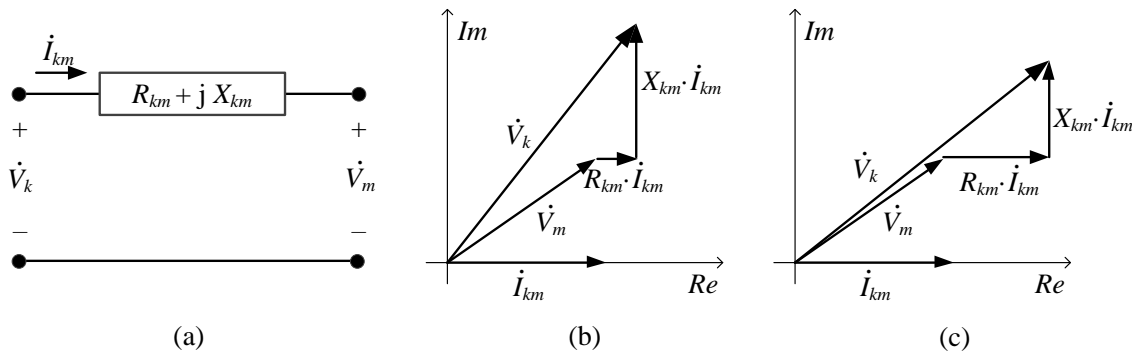


Figure 2.1: Model of a distribution circuit (a); phase diagram considering low R/X ratio; (c) phase diagram considering high R/X ratio

Part (a) of Figure 2.1 depicts a simple series-impedance model of a fictitious distribution circuit, in which the line current lags the voltages at the two extremities. Parts (b) and (c) indicate phase diagrams, with a higher R/X ratio considered for the circuit of part (c). To facilitate the discussion, the modulus of the branch impedance,

$|R_{km} + j \cdot X_{km}|$, is kept approximately constant while varying the R/X ratio from Figure 2.1.(b) to Figure 2.1.(c).

By comparing the phase diagrams, it becomes clear that a higher R/X ratio results in a lower angular difference between the complex voltages \dot{V}_k and \dot{V}_m . This illustrative analysis alludes to the fact that, due to the high R/X ratios, the angular differences between complex voltages of buses of a distribution network usually do not display values as high as those from transmission systems.

Figure 2.1 is merely illustrative, and the configuration of the diagrams would vary if the phase angle difference between the voltage \dot{V}_m and the line current \dot{I}_{km} were modified. One of the factors that affect the phase angle between bus voltages and branch currents is the power factor of bus injections. At this point, the reader shall keep in mind that there are usually incentives for customers connected to distribution systems to keep the power factor of their loads within relatively narrow intervals – e.g., the Brazilian regulation [68] prescribes that the power factor of loads connected to distribution systems at all voltage levels below 230 kV shall be kept within the interval $[0.92_{\text{lagging}}, 0.92_{\text{leading}}]$. The fact that load power factors are usually kept close to unitary values basically contributes to keeping the angular differences among complex voltages of buses of the distribution network at low values.

Thus, if any given bus within the distribution system or at its frontier (e.g., the bus that represents the high-voltage side of the step-down transformer at an interface with the subtransmission or transmission system) is chosen to be the angular reference bus, and a reference angle of $\theta^{ref} = 0^\circ$ is attributed to it, the voltage angles of all buses in the distribution network will usually vary within a narrow interval around zero. The reader shall keep this in mind, as this fact will be relevant for the presentation of the MILP reformulation of the ACOPF in distribution systems, in chapter 4.

2.1.3 Radiality constraints and reconfiguration

As of this writing, distribution systems are predominantly operated radially, as the radial configuration allows that adequate protection coordination can be achieved even if more economical protection equipment is used – e.g., the protection system may be built mainly upon fuses, which are not only economical but also comparatively reliable in interrupting fault currents [6]. There are, however, distribution systems that

are operated as meshed networks [6], [69]. In fact, meshed operation may be economical under strict reliability requirements, and it has been argued that, under specific conditions, it may be adequate to facilitate the penetration of distributed generation [70].

Many distribution systems that are operated radially are meshed *in design* – this meaning that there are switches that may be opened or closed to reconfigure the system both in response to a disturbance (e.g., allowing the isolation of a fault) or to enhance operating efficiency (e.g., with respect to ohmic losses) [64]. Evidently, in distribution systems for which radial operation is required in order to achieve protection coordination, any reconfiguration of the network shall comply with radiality constraints.

2.1.4 Unbalance between phases

Distribution systems may be subject to unbalanced conditions due to structural and operational factors [71]. Structural unbalanced relates to aspects such as the existence of single-phase or two-phase circuits (mainly in secondary systems), incomplete transposition of three-phase circuits, asymmetrical wiring of transformers, etc. Operational unbalance is that associated with the uneven distribution of single-phase and two-phase loads within the network, and to unbalanced three-phase loads [71]. Unbalanced operation in distribution systems may lead to increased losses, limit transformer loading and bring additional problems with respect to voltage control [71].

The assessment of the impacts of unbalanced operation in power flow simulations requires the use of an unbalanced three-phase model, allowing the representation of different electrical parameters for each phase of the circuits, as well as permitting the modeling of unbalanced loads. It should be noted, however, that unbalance between phases in the primary distribution system (medium voltage) is less significant than that of the secondary distribution system (low voltage), and that, within the primary distribution system, unbalance is less significant in feeders (usually three-phase circuits) than laterals [72].

The ACOPF formulation proposed in this dissertation is based on the equivalent single-phase model for balanced three-phase electrical systems, and does not apply to unbalanced distribution systems. Its primary applicability is therefore to the primary

feeder system⁴. As indicated in section 6, the extension of the proposed MILP formulation of the ACOPF model to unbalanced three-phase systems is a possible topic for future work.

2.2 The ACOPF for distribution systems

This section is dedicated to the presentation of the non-linear version of the ACOPF problem for distribution systems⁵, with focus on mathematical modeling.

Section 2.2.1 introduces the constraints of the ACOPF, through which the electrical behavior of the network and of bus injections is modeled. Constraints related to equipment operating limits are also presented. Objective functions associated with selected applications of the ACOPF for distribution system operations and expansion planning are presented in section 2.2.2.

Though discrete decisions are briefly mentioned in the following sections, their full mathematical formulation is presented only in chapter 4. Nonetheless, the reference to discrete decisions in this section will allow the reader to notice that the ACOPF for distribution system operations and expansion planning applications is a non-convex, mixed-integer non-linear programming problem (MINLP). Techniques for the reformulation of such problems as mixed-integer linear programs will be presented in chapter 3.

The nomenclature used in this and other chapters of this dissertation has been presented at a specific section of this document, before the introductory chapter.

2.2.1 Constraints: modeling electrical behavior and enforcing operating limits

The formulation presented below is based on expressing complex variables in rectangular coordinates (real and imaginary components, as opposed to angles and

⁴ It is worth mentioning that, for many of the applications of interest to distribution systems engineers, analyses restricted to the primary feeder system are sufficient – e.g., switchable elements are usually restricted to the feeder system, meaning that reconfiguration studies executed with models restricted to this system will usually lead to satisfactorily accurate results.

⁵ The reader will notice that the formulation presented here is not yet the proposed MILP reformulation of the ACOPF, which will be presented only in chapter 4.

magnitudes), and utilizing voltages and *currents* (as opposed to voltages and *power quantities*) to describe Kirchhoff's laws. The motivation for these modeling choices will be presented further in this document.

2.2.1.1 Kirchhoff's Laws

Equations (1) and (2) model Kirchhoff's Current Law for all buses in the distribution system:

$$I_{d,k}^{re} + \sum_{m \in \Omega_k} I_{km}^{re} = I_{g,k}^{re} + \sum_{m \in \Omega_k} I_{mk}^{re} \quad , \forall k \in \Omega_B \quad (1)$$

$$I_{d,k}^{im} + \sum_{m \in \Omega_k} I_{km}^{im} = I_{g,k}^{im} + \sum_{m \in \Omega_k} I_{mk}^{im} \quad , \forall k \in \Omega_B \quad (2)$$

where:

$k ; m$	Indices for buses of the distribution system;
Ω_B	Set of all buses in the distribution system;
Ω_k	Set of buses directly connected to bus k ;
$I_{d,k}^{re}$	Real component of current demanded by load connected to bus k ;
$I_{d,k}^{im}$	Imaginary component of current demanded by load connected to bus k ;
$I_{g,k}^{re}$	Real component of current generated by generator connected to bus k ;
$I_{g,k}^{im}$	Imaginary component of current generated by generator connected to bus k ;
I_{km}^{re}	Real component of current flowing through the branch connecting buses k and m , from bus k to bus m ;
I_{km}^{im}	Imaginary component of current flowing through the branch connecting buses k and m , from bus k to bus m .

The decision variables in equations (1) and (2) are $I_{d,k}^{re}$, $I_{d,k}^{im}$, $I_{g,k}^{re}$, $I_{g,k}^{im}$, I_{km}^{re} and I_{km}^{im} (continuous decision variables, free in signal).

Equations (3) and (4) model Kirchhoff's Voltage Law for all branches in the distribution system:

$$V_k^{re} - V_m^{re} = I_{km}^{re} \cdot R_{km} - I_{km}^{im} \cdot X_{km} \quad , \forall km \in \Psi_C \quad (3)$$

$$V_k^{im} - V_m^{im} = I_{km}^{re} \cdot X_{km} + I_{km}^{im} \cdot R_{km} \quad , \forall km \in \Psi_C \quad (4)$$

where:

km	Index for branches of the distribution system;
Ψ_C	Set of all branches in the distribution system;
V_k^{re}	Real component of voltage at bus k ;
V_k^{im}	Imaginary component of voltage at bus k .

At this point, a few words on the notation employed for in the above definitions are in order. The set Ψ_C is considered to be a set of ordered pairs, and the first and second entries of each ordered pair correspond to the *from* and *to* buses of a branch in the distribution system. Thus, the element $\langle k, m \rangle$ of the set $\Psi_C = \{\langle 1,2 \rangle, \langle 1,3 \rangle \dots \langle k, m \rangle\}$ indicates the distribution circuit that connects bus k to bus m . For the sake of conciseness of notation, we refer to $\langle k, m \rangle$ simply as km .

The decision variables in equations (3) and (4) that have not yet been identified are V_k^{re} (continuous, non-negative⁶) and V_k^{im} (continuous, free in sign).

The constraints represented by equations (1) to (4) are linear, and can therefore be readily represented in linear or mixed-integer linear programs. In fact, the linearity of the equations that describe Kirchoff's laws is one of the reasons for employing a rectangular formulation for the power flow equations, with basis on voltages and currents (as opposed to voltages and power quantities) values.

For *switchable* branches in the *off* state or for *candidate* branches (candidates for distribution system expansion) that have not been constructed, these constraints must be relaxed. This will be discussed in chapter 4.

2.2.1.2 Generators

As Kirchoff's laws have been formulated with basis on voltages and currents (as opposed to voltages and power quantities), it is necessary to obtain the (voltage-dependent) values of $I_{g,k}^{re}$ and $I_{g,k}^{im}$ for all generators in the system. This is done with help of equations (5) and (6):

⁶ For the typical bus voltage angles verified in distribution systems (considering that the angular reference bus is within the distribution system and that the reference angle is zero), V_k^{re} may be characterized as a non-negative decision variable.

$$I_{g,k}^{re} = \xi_k \cdot g_k^P + \zeta_k \cdot g_k^Q, \quad \forall k \in \Omega_{GEN} \quad (5)$$

$$I_{g,k}^{im} = \zeta_k \cdot g_k^P - \xi_k \cdot g_k^Q, \quad \forall k \in \Omega_{GEN} \quad (6)$$

where:

Ω_{GEN} Set of all buses to which generators are connected;

g_k^P Active power generated by generator at bus k ;

g_k^Q Reactive power generated by generator at bus k ;

$\xi_k ; \zeta_k$ Auxiliary decision variables, defined below.

The auxiliary variables ξ_k and ζ_k are dependent on the real and imaginary components of the voltage at bus k , as shown below:

$$\xi_k = V_k^{re} / (V_k^{re2} + V_k^{im2}), \quad \forall k \in \{\Omega_{GEN} \cup \Omega_{PCTE}\} \quad (7)$$

$$\zeta_k = V_k^{im} / (V_k^{re2} + V_k^{im2}), \quad \forall k \in \{\Omega_{GEN} \cup \Omega_{PCTE}\} \quad (8)$$

The decision variables in equations (5) to (8) that have not yet been identified are ξ_k (continuous, non-negative⁷) and ζ_k (continuous, free in signal). It is assumed that the reactive power output of all generators in the system is controllable, and g_k^Q is thus a (continuous) decision variable. Also, if the active power output of the generator at bus k is controllable, g_k^P is a (continuous) decision variable. Whether g_k^P and g_k^Q are non-negative or free in signal will depend on the bounds defined as inputs for the ACOPF – for typical applications, g_k^P will be non-negative and g_k^Q will be free in signal.

The reader will notice that the constraints specified in equations (7) and (8) are enforced not only for the buses pertaining to Ω_{GEN} , but also for those in the set Ω_{PCTE} . This latter set will be defined in section 2.2.1.3.1.

The non-linear nature of the constraints represented by equations (7) and (8) is evident.

⁷ For the typical bus voltage angles verified in distribution systems (considering that the angular reference bus is within the distribution system and that the reference angle is zero), ξ_k may be characterized as a non-negative decision variable.

2.2.1.3 Loads

Analogously to what has been done for the current injections from generators, it is necessary to define constraints through which the currents $I_{d,k}^{re}$ and $I_{d,k}^{im}$ will be obtained, for all loads in the system. Those constraints are presented in the following sections, for *constant-power*, *constant-current* and *constant-impedance* loads. These are the three basic components of the widely employed static load model known as *ZIP model* [73]. For the sake of conciseness of presentation, the equations presented below consider that the load at any given bus is modeled as *purely of the constant-power type*, *purely of the constant-current type* or *purely of the constant-impedance type*. Still, the modification of the equations to account for any affine combination of these types of loads is trivial.

2.2.1.3.1 Constant-power loads

The currents demanded by constant-power loads are obtained with help of equations (9) and (10):

$$I_{d,k}^{re} = \xi_k \cdot d_k^P + \zeta_k \cdot d_k^Q, \forall k \in \Omega_{PCTE} \quad (9)$$

$$I_{d,k}^{im} = \zeta_k \cdot d_k^P - \xi_k \cdot d_k^Q, \forall k \in \Omega_{PCTE} \quad (10)$$

where:

Ω_{PCTE} Set of all buses to which constant-power loads are connected;

d_k^P Nominal value of active power demanded by load at bus k ;

d_k^Q Nominal value of reactive power demanded by load at bus k .

The auxiliary variables ξ_k and ζ_k have already been defined through equations (7) and (8). Constraints (9) and (10) are linear.

2.2.1.3.2 Constant-current loads

Keeping in mind that *constant-current* loads are characterized by the linear variation of the demanded power with respect to bus voltage magnitude [73], the currents demanded by these loads can be obtained with help of equations (11) and (12):

$$I_{d,k}^{re} = \eta_k \cdot d_k^P + \kappa_k \cdot d_k^Q, \forall k \in \Omega_{ICTE} \quad (11)$$

$$I_{d,k}^{im} = \kappa_k \cdot d_k^P - \eta_k \cdot d_k^Q, \forall k \in \Omega_{ICTE} \quad (12)$$

where:

Ω_{ICTE} Set of all buses to which constant-current loads are connected;

$\eta_k ; \kappa_k$ Auxiliary decision variables, defined below.

The auxiliary variables η_k and κ_k are dependent on the real and imaginary components of the voltage at bus k , and defined as:

$$\eta_k = V_k^{re} / \sqrt{V_k^{re2} + V_k^{im2}}, \forall k \in \Omega_{ICTE} \quad (13)$$

$$\kappa_k = V_k^{im} / \sqrt{V_k^{re2} + V_k^{im2}}, \forall k \in \Omega_{ICTE} \quad (14)$$

The decision variables in equations (11) to (14) that have not yet been identified are η_k (continuous, non-negative⁸) and κ_k (continuous, free in signal).

Constraints (13) and (14) are clearly non-linear.

2.2.1.3.3 Constant-impedance loads

The currents demanded by constant-impedance loads can be obtained with help of the following constraints:

⁸ For the typical bus voltage angles verified in distribution systems (considering that the angular reference bus is within the distribution system and that the reference angle is zero), η_k may be characterized as a non-negative decision variable.

$$I_{d,k}^{re} = V_k^{re} \cdot \frac{R_k^l}{|Z_k^l|^2} + V_k^{im} \cdot \frac{X_k^l}{|Z_k^l|^2}, \forall k \in \Omega_{ZCTE} \quad (15)$$

$$I_{d,k}^{im} = V_k^{im} \cdot \frac{R_k^l}{|Z_k^l|^2} - V_k^{re} \cdot \frac{X_k^l}{|Z_k^l|^2}, \forall k \in \Omega_{ZCTE} \quad (16)$$

where:

Ω_{ZCTE} Set of all buses to which constant-impedance loads are connected;

R_k^l Resistance of constant-impedance load at bus k ;

X_k^l Reactance of constant-impedance load at bus k ;

Z_k^l Impedance of constant-impedance load at bus k .

All decision variables in constraints (15) and (16) have been previously identified. The reader will notice that (15) and (16) correspond to linear constraints.

At this point, it is worth recalling that the *nominal value* of the load associated with constant-impedance loads – i.e., the value of the load at the voltage of $(1\angle 0^\circ)$ p.u. – is given by:

$$d_k^P = \frac{R_k^l}{|Z_k^l|^2} \quad ; \quad d_k^Q = \frac{X_k^l}{|Z_k^l|^2}, \forall k \in \Omega_{ZCTE} \quad (17)$$

2.2.1.4 Operating limits

2.2.1.4.1 Bounds on bus voltage magnitudes

Constraints that ensure that bus voltage magnitudes are kept within admissible limits are presented below:

$$V_k = \sqrt{V_k^{re2} + V_k^{im2}}, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (18)$$

$$\underline{V}_k \leq V_k \leq \bar{V}_k, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (19)$$

where:

Ω_{REF} Set of voltage reference buses in the system;

V_k Magnitude of voltage at bus k ;

$\underline{V}_k ; \bar{V}_k$ Lower and upper bounds for magnitude of voltage at bus k .

The decision variable in equations (18) and (19) that has not yet been identified is V_k (continuous, non-negative).

The operator \setminus indicates set difference – i.e., $A \setminus B = \{x \in A \text{ and } x \notin B\}$, where A and B are sets. The reader will notice that constraints (18) and (19) are not enforced for the set of voltage reference buses in the distribution system, as discussed in in section 2.2.1.5.

Constraint (18) is evidently non-linear.

2.2.1.4.2 Bounds on branch currents

At this point, it is important to recall that thermal loading limits of transmission lines are actually related to current loading, despite the fact that, mainly in applications of the ACOPF to transmission systems, it is common to represent these thermal loading constraints approximately as bounds on apparent power flows.

The following constraints ensure that branch current magnitudes are kept within admissible limits:

$$I_{km} = \sqrt{I_{km}^{re\ 2} + I_{km}^{im\ 2}} \quad , \forall k \in \Psi_C \quad (20)$$

$$I_{km} \leq \bar{I}_{km} \quad , \forall k \in \Psi_C \quad (21)$$

where:

I_{km} Magnitude of current flowing through branch km ;

\bar{I}_{km} Upper bound for magnitude of current flowing through branch km .

The decision variable in equations (20) and (21) that has not yet been identified is I_{km} (continuous, non-negative).

The reader will notice that (20) is a non-linear constraint.

2.2.1.4.3 Bounds on active and reactive power output of generators

Constraints that ensure that the active and reactive power output of generators are kept within the admissible ranges are presented below:

$$\underline{g}_k^P \leq g_k^P \leq \overline{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (22)$$

$$\underline{g}_k^Q \leq g_k^Q \leq \overline{g}_k^Q, \forall k \in \Omega_{GEN} \quad (23)$$

where:

Ω_{CTRPQ} Set of buses to which generators with control over the output of active and reactive power connect;

$\underline{g}_k^P ; \overline{g}_k^P$ Lower and upper bounds for active power output of generator at bus k ;

$\underline{g}_k^Q ; \overline{g}_k^Q$ Lower and upper bounds for reactive power output of generator at bus k .

Equations (22) and (23) correspond to linear constraints.

2.2.1.5 Voltage reference buses

If the representation of more than one islanded system in a single ACOPF problem is required, it is necessary to define one (and only one) angular reference bus for each island. For this reason, we refer to the definition of voltage reference *buses* (plural emphasized), which pertain to the set Ω_{REF} . Obviously, for any specific ACOPF application that requires the representation of a single electrical island, the cardinality of the set of reference buses will be $|\Omega_{REF}| = 1$.

The real and imaginary components of the complex voltage at the angular reference bus may be specified with help of the following constraints:

$$V_k^{re} = V_k^{ref} \cdot \cos\theta_k^{ref}, \forall k \in \Omega_{REF} \quad (24)$$

$$V_k^{im} = V_k^{ref} \cdot \sin\theta_k^{ref}, \forall k \in \Omega_{REF} \quad (25)$$

where:

V_k^{ref} Fixed voltage magnitude of reference bus, an input parameter for the ACOPF;

$\cos\theta_k^{ref}$; $\sin\theta_k^{ref}$
Cosine and sine of reference angle for reference bus.

As the voltage magnitudes of reference buses are fixed, it is not necessary to enforce constraints (18) and (19) for them, hence the previous definition of these constraints.

If, alternatively, the voltage magnitudes at the reference buses are to be considered decision variables in any specific application, equations (24) and (25) should be substituted for the following constraints:

$$V_k^{re} = V_k \cdot \cos\theta_k^{ref} \quad , \forall k \in \Omega_{REF} \quad (26)$$

$$V_k^{im} = V_k \cdot \sin\theta_k^{ref} \quad , \forall k \in \Omega_{REF} \quad (27)$$

where, as previously stated, V_k is a continuous decision variable. In this case, constraint (19) shall also be enforced for the set of reference buses.

Constraints (24) to (27) are linear.

2.2.1.6 Slack buses and buses without generators and/or loads

The following set of constraints ensures that the load currents of all buses to which no loads connect are set to zero:

$$I_{d,k}^{im} = I_{d,k}^{im} = 0 \quad , \forall k \in \{\Omega_B \setminus \Omega_{LOAD}\} \quad (28)$$

where:

Ω_{LOAD} Set of buses to which loads connect, $\Omega_{LOAD} = \{\Omega_{PCTE} \cup \Omega_{ICTE} \cup \Omega_{ZCTE}\}$.

The following set of constraints ensures that the generator currents of all buses to which no generators connect are set to zero:

$$I_{g,k}^{im} = I_{g,k}^{im} = 0 \quad , \forall k \in \{\Omega_B \setminus \{\Omega_{GEN} \cup \Omega_{SLACK}\}\} \quad (29)$$

where:

Ω_{SLACK} Set of all slack buses in the system.

The reader will notice that (29) ensures that the generator currents $I_{g,k}^{im}$ and $I_{g,k}^{im}$ may assume any given value for the buses in set Ω_{SLACK} . It is important to emphasize that, in the ACOPF problem, it is not necessary that slack buses are defined – an ACOPF problem without slack buses is potentially feasible whenever the generating capacity within the system is sufficient to supply its load and cover ohmic losses. However, for some specific applications, it may be in the interest of the distribution system engineer to define slack buses, and in these cases Ω_{SLACK} will be a nonempty set.

It should also be emphasized that it is not necessary that the sets Ω_{SLACK} and Ω_{REF} coincide – i.e., a slack bus may or may not be a voltage reference bus, and a voltage reference bus may or may not be a slack bus.

2.2.1.7 Radiality constraints

The formulation of radiality constraints demands the use of binary decision variables, and will therefore be presented only in chapter 4. For now, it suffices to indicate that radiality constraints will ensure that each *active bus* in every island of the system will be connected to the *root bus* of that island via one and only one electrical path, with no loops. The *root bus* is that from which the radial network originates, and, for most practical applications, this will be the bus at the interface of the distribution system with the transmission or subtransmission system.

It is worth recalling that, as stated in section 2.1.3, there are distribution systems that are operated in a meshed, and not a radial, fashion. The ACOPF formulation proposed in this dissertation is valid both for meshed and for radial systems – the difference is that, if radiality is required, a specific set of radiality constraints (that will be indicated in chapter 4) must be enforced.

2.2.2 Objective functions for selected distribution system operations and expansion planning applications

In this section, the mathematical formulation of objective functions associated with selected applications of the ACOPF for distribution system operations and expansion planning is presented. The equations of this section are potentially non-linear, and their MILP reformulation will be presented in chapter 4. Though applications involving discrete decisions are preliminarily presented here (so that the reader can have a better comprehension of the full problem to which the linearization and convexification techniques presented in chapter 3 will be applied), their full mathematical formulation will be shown only in chapter 4.

Each of the following subsections will begin with the mathematical formulation of an objective function, such as minimization of generation costs, minimization of costs of losses, minimization of load shedding costs, etc. These may be also interpreted as *modules* of a composite objective function – e.g., a given distribution system operations planning application may require the simultaneous minimization of losses *and* of load shedding costs. An enumeration of practical applications of the presented objective functions will follow the mathematical formulation in each subsection. This enumeration aims not at being exhaustive, but only at illustrating the flexibility of the proposed formulation of the ACOPF problem.

The operations and expansion planning applications presented in this chapter involve the evaluation of a single operating point of the distribution grid, which constrains the universe of treatable problems to deterministic, single-stage applications. It is worth mentioning, however, that both the non-linear formulation presented in this chapter and the MILP reformulation presented in chapter 4 may be employed in applications in which more than one operating point is evaluated. Thus, it is theoretically possible to treat stochastic and multi-stage problems – naturally, at the cost of augmented computational requirements –, even though this topic is has not yet been subject to research.

2.2.2.1 Minimization of costs of load shedding

In case of contingencies or disturbances that affect the distribution network, load shedding may be adopted as a last-resource remedial action. As of this writing, most distribution utilities implement load shedding via controlled de-energization of entire segments of the distribution network, mainly by maneuvering switches in the primary distribution feeder system. With this implementation, each load in the network will be either completely de-energized (shed) or will not experience any load shedding at all – thus, a representation of the *discrete* nature of the decision to de-energize of each load is required.

Assuming that the costs of load shedding are proportional to the *nominal value* of the loads in the network, the following formulation may be defined:

$$z^{SHED} = \min\{\sum_{k \in \Omega_{LOAD}} c_k^{SHED} \cdot d_k^P \cdot \rho_k\} \quad (30)$$

where:

- z^{SHED} Value of the objective function modeling the (minimization of) load shedding costs; z^{SHED} may also be used as a parcel of a composite objective function;
- Ω_{LOAD} Set of all buses to which loads are connected, defined as $\Omega_{LOAD} = \{\Omega_{PCTE} \cup \Omega_{ICTE} \cup \Omega_{ZCTE}\}$;
- c_k^{SHED} Cost coefficient associated with load shedding at bus k ;
- ρ_k Binary decision variable that indicates if load at bus k is shed ($\rho_k = 1$ indicates that load is shed).

This discrete modeling of load shedding demands the modification of some of the constraints defined in section 2.2.1. As the modification of these constraints involves the use of discrete decision variables, it will be discussed in details only in chapter 4.

Future technological advancements may facilitate the widespread employment of other load shedding mechanisms, including those in which the utility decides on *what parcel of the load* to curtail in each bus – i.e., the amount of load shedding in each bus would be a *continuous decision variable*. If such mechanisms are to be considered in an

ACOPF application, d_k^P should be modeled as a continuous decision variable, and the following objective function would be employed:

$$z^{SHDC} = \min\{\sum_{k \in \Omega_{LOAD}} c_k^{SHED} \cdot (d_k^{P,ref} - d_k^P)\} \quad (31)$$

where:

z^{SHDC} Value of the objective function modeling the (minimization of) load shedding costs, considering the case in which d_k^P is modeled as a continuous decision variable; z^{SHDC} may also be used as a parcel of a composite objective function;

$d_k^{P,ref}$ Reference value (value with no load shedding) of the active load at bus k .

This work will focus on the former formulation of the objective function (z^{SHED} , as opposed to z^{SHDC}), as it currently corresponds to the more common practice for the implementation of emergency, last resource load shedding actions. Thus, for all equations presented in this dissertation, with the exception of equation (31), d_k^P is a parameter (the nominal value of the active load at bus k).

The minimization of load shedding costs may compose the objective function in a wide range of applications in distribution operations and expansion planning, such as:

- Elaborations of contingency plans;
- Reliability studies (which would require the evaluation of more than one operating point);
- Comparison of alternatives and estimation of added value of reinforcements, in the context of system expansion planning.

2.2.2.2 *Minimization of costs of curtailment of non-controllable generation*

The distribution utility may not have full control over the output of some of the distributed generators in its network, either due to these generators being located at the consumer side of the meter (assuming that there is no centralized dispatch mechanism in force) or to the fact that they rely on primary energy sources that are essentially non-controllable, as in the case of solar photovoltaic panels. Depending on specific incentive

mechanisms for distributed generation prescribed by regulation in each jurisdiction, it may be in the utility's interest to minimize the curtailment of the output of these kinds of generators – e.g., if penalties or monetary compensations are imposed in case renewable distributed generation is curtailed.

Assuming that the active power output of the distributed generation is not controllable, and that the only response to short-term violation of operating limits caused by these generation is their disconnection from the grid⁹, the objective function for the minimization of costs of curtailment of non-controllable distributed generation may be formulated as follows:

$$z^{GCRT} = \min \left\{ \sum_{k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\}} c_k^{CURT} \cdot g_k^P \cdot \tau_k \right\} \quad (32)$$

where:

- z^{GCRT} Value of the objective function modeling the (minimization of) non-controllable generation curtailment costs; z^{GCRT} may also be used as a parcel of a composite objective function;
- Ω_{CTRQ} Set of buses to which generators with non-controllable active power output (reactive power output assumed to be controllable) connect, defined as $\Omega_{CTRQ} = \Omega_{GEN} \setminus \Omega_{TRPQ}$;
- Ω_{CURT} Set of buses to which curtailable generators connect;
- c_k^{CURT} Cost coefficient associated with curtailment of generator at bus k ;
- τ_k Binary variable that indicates if generator is curtailed ($\tau_k = 1$ indicates that generator is curtailed).

The minimization of the costs of curtailment of non-controllable generation may compose the objective function for applications such as:

- Determination of the maximum penetration of distribution generation;
- Distribution system expansion and operations planning under explicit modeling of generation curtailment costs.

⁹ Alternatively, we may think of the discrete decision to *curtail* a generator, in the context of medium-term planning, as an indication of the need to prohibit or postpone its grid connection until future reinforcements ensure technical feasibility. This will be further explored in section 5.2.2.

2.2.2.3 Minimization of generation costs

Assuming that generators with non-controllable active power output have null variable operation costs (or at least null costs perceived by the distribution utility), the objective function for the minimization of variable generation costs within the distribution system may be formulated as follows:

$$z^{GEN} = \min\left\{\sum_{k \in \Omega_{CTRPQ}} c_k^{GEN} \cdot g_k^P\right\} \quad (33)$$

where:

z^{GEN} Value of the objective function modeling the (minimization of) variable generation costs; z^{GEN} may also be used as a parcel of a composite objective function;

c_k^{GEN} Cost coefficient associated with generation with controllable active power output at bus k .

The minimization of variable generation costs may compose the objective function for applications such as the economic dispatch of generation resources within the distribution system.

2.2.2.4 Minimization of costs of power imports

It may be necessary to model the costs associated with power imports from an external network (the transmission system or even other distribution systems) in a variety of operations or expansion planning applications. One possible way of doing that is by modeling power imports as the output of a *virtual generator*, and employing the objective function defined in section 2.2.2.3.

Another modeling choice would be to represent the bus at the interface with the external network as a *slack bus which is also a reference bus* (implicitly considering this bus as an idealized voltage source), and to associate costs to the infeed of active power at this bus. Considering this, and assuming the most general case in which the reference voltage magnitudes of all buses at the interface with the external network are

considered as decision variables, the formulation of the objective function for the minimization of the costs of power imports would be:

$$z^{IMPR} = \min\{\sum_{k \in \Omega_{ITFC}} c_k^{IMPORT} \cdot V_k \cdot (\cos\theta_k^{ref} \cdot I_{g,k}^{re} + \sin\theta_k^{ref} \cdot I_{g,k}^{im})\} \quad (34)$$

where:

Ω_{ITFC} Set of buses at the interface of the internal network with the external network, considered to be defined as $\Omega_{ITFC} = \Omega_{SLACK} \cap \Omega_{REF}$;

z^{IMPR} Value of the objective function modeling the (minimization of) costs of power imports from an external network; z^{IMPR} may also be used as a parcel of a composite objective function;

c_k^{IMPORT} Cost coefficient associated with imports from the external network, at the interface represented as the slack bus k .

It should be kept in mind that, as pointed out in section 2.2.1.5, it is necessary to define one (and only one) angular reference bus for each island of the distribution system to be simulated. Thus, for most conceivable practical applications, the voltage angle of all buses in angle of Ω_{ITFC} may be set to $\theta_k^{ref} = 0^\circ$, without loss of generality. After that, equation (34) may be rewritten as:

$$z^{IMPR} = \min\{\sum_{k \in \Omega_{ITFC}} c_k^{IMPORT} \cdot V_k \cdot I_{g,k}^{re}\} \quad (35)$$

If, besides all modeling assumptions considered so far, the reference voltage magnitude of all buses in Ω_{ITFC} is fixed at any arbitrary value V_k^{ref} (an input parameter of the ACOPT), the last expression may be rewritten as:

$$z^{IMPR} = \min\{\sum_{k \in \Omega_{ITFC}} c_k^{IMPORT} \cdot V_k^{ref} \cdot I_{g,k}^{re}\} \quad (36)$$

The minimization of costs associated with power imports from an external network may be employed in application as:

- Least-cost operations and expansion planning studies;

- Determination of the amount of imports to be contracted at the interface with external networks.

2.2.2.5 Minimization of costs of ohmic losses

The total ohmic losses within a given distribution system may be calculated either by summing the losses of each individual circuit, or by determining the difference between the total active power injected into the distribution network and the total active power consumed. The latter option is considered for the definition of the following objective function:

$$\begin{aligned}
z^{LOSS} = \min \{ & c^{LOSS} \cdot \{ \sum_{k \in \Omega_{ITFC}} V_k^{ref} \cdot I_{g,k}^{re} + \sum_{k \in \Omega_{CTRPQ}} g_k^P \\
& + \sum_{k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\}} g_k^P + \sum_{k \in \Omega_{CURT}} g_k^P \cdot (1 - \tau_k) \\
& - [\sum_{k \in \{\Omega_{PCTE} \cap \Omega_{NSHD}\}} d_k^P + \sum_{k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\}} d_k^P \cdot (1 - \rho_k)] \\
& - [\sum_{k \in \{\Omega_{ICTE} \cap \Omega_{NSHD}\}} V_k \cdot d_k^P + \sum_{k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\}} V_k \cdot d_k^P \cdot (1 - \rho_k)] \\
& - \left[\sum_{k \in \{\Omega_{ZCTE} \cap \Omega_{NSHD}\}} V_k^2 \cdot \frac{R_k^l}{|Z_k^l|^2} + \sum_{k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\}} V_k^2 \cdot \frac{R_k^l}{|Z_k^l|^2} \cdot (1 - \rho_k) \right] \} \}
\end{aligned} \tag{37}$$

where:

z^{LOSS} Value of the objective function modeling the (minimization of) costs of ohmic losses; z^{LOSS} may also be used as a parcel of a composite objective function;

c^{LOSS} Cost coefficient associated with ohmic losses.

The reader will notice that the first summation at the right portion of equation (37) corresponds to the power imported from external networks. For the sake of conciseness of presentation, we consider the case in which the voltages of all buses in Ω_{ITFC} are fixed at $(V_k^{ref} \angle 0^\circ)$ p.u. However, the other (more general) cases described in section 2.2.2.4 may also be considered while formulating this objective function.

Yet, even under consideration of the simplest case for the imports from external networks, equation (37) is obviously non-linear. Keeping in mind that d_k^P is a parameter

of the ACOFP, it is clear that the non-linearities are associated with the terms $V_k \cdot \rho_k$, V_k^2 and $V_k^2 \cdot \rho_k$.

The cost coefficient c^{LOSS} may be set to unity if the value of the objective function is to be expressed in MW (or p.u.) rather than in monetary units (\$). In fact, this cost coefficient may be manipulated according to the requirements of the specific application – e.g., if the evaluated operating point is deemed representative of any given time interval, the costs coefficient may be determined by the multiplication of the duration of the interval in hours and the cost of losses in \$/MWh. The same consideration basically applies to all cost coefficients presented so far.

The minimization of (the costs of) ohmic losses may compose the objective function in a wide range of applications in operations and expansion planning, such as:

- Network reconfiguration studies;
- Integrated voltage/VAr control planning;
- Planning of network reinforcements (current carrying-facilities);
- Capacitor placement and sizing planning;
- Planning of placement and control of distributed generation.

2.2.2.6 *Minimization of costs of reinforcements to the distribution system*

Expansion planning applications require the determination of the optimal set of reinforcements to the distribution system, usually with focus on new circuits, substations, and equipment for reactive power support. According to the planning objectives of a given utility, the objective function of the planning problem may include different components – one of the most important being the costs of reinforcements. The objective function for the minimization of the costs of network reinforcements is indicated below:

$$z^{REIN} = \min\{\sum_{km \in \Psi_{CD}} c_{km}^{CONST} \cdot \sigma_{km}\} \quad (38)$$

where:

z^{REIN} Value of the objective function modeling the (minimization of) costs of reinforcements to the distribution system; z^{REIN} may also be used as a parcel of a composite objective function;

Ψ_{CD}	Set of circuits that represent candidate reinforcements to current-carrying facilities;
c_{km}^{CONST}	Cost associated with construction of reinforcement represented by circuit km ;
σ_{km}	Binary variable that represents decision of constructing the reinforcement represented by km ($\sigma_{km} = 1$ indicates that reinforcement is built).

At this point, it is worth mentioning that a fictitious candidate circuit may be used to model either a candidate substation or candidate reactive power support equipment. For that, it suffices to ensure that the bus corresponding to the candidate substation/equipment is only included into the network if the fictitious candidate circuit is built (which can be done by manipulating the equivalent network topology), and set the value of the impedance of the candidate circuit in order to ensure that its inclusion will not materially affect the solution of the ACOPF problem. Naturally, the costs associated with the candidate substation/equipment would be represented via the c_{km}^{CONST} of the candidate circuit.

Obviously, it is necessary to ensure that the constraints associated with candidate circuits that are not built are relaxed, in the formulation of the ACOPF. As this requires discrete decision variables, this matter will be discussed further only in chapter 4.

2.2.2.7 *Minimization of costs of capacitor placement*

The capacitor placement problem involves determining the optimal location and sizing of capacitors to be added to the distribution network.

One option to account for the capacitor placement costs while determining the optimal network expansion plan is to employ the same basic formulation described in section 2.2.2.6, and then represent the candidate capacitors as a purely reactive (and capacitive) load at a candidate bus that is connected to the remainder of the system via a fictitious, low-impedance circuit. In this case, the costs of the candidate capacitors would be attributed to the candidate, fictitious circuits.

An alternative for factoring capacitor placement costs into the objective function of an ACOPF is to consider the capacitors as a purely reactive, “curtailable” load, and then associate the costs of installing the capacitor to the change of status of this reactive

load from *inactive* to *active*. Mathematically, this corresponds to the following objective function:

$$z^{CAPL} = \min\{\sum_{k \in \Omega_{CAP}} c_k^{CAPL} \cdot (1 - \rho_k)\} \quad (39)$$

where:

- z^{CAPL} Value of the objective function modeling the (minimization of) capacitor placement costs; z^{CAPL} may also be used as a parcel of a composite objective function;
- Ω_{CAP} Set of buses with candidate capacitors;
- c_k^{CAPL} Cost coefficient associated with the placement of the candidate capacitor (purely reactive, capacitive load) at bus k ;
- ρ_k Binary variable that indicates if the capacitor (purely reactive, capacitive load) is connected to the system ($\rho_k = 0$ indicates that capacitor was installed and is connected to the system).

As previously mentioned, costs of capacitor placement are considered within distribution system expansion planning applications.

2.2.2.8 *Minimization of circuit switching costs*

It is not customary to consider switching costs in applications of distribution system operations and planning – normally, the costs of switching actions are considered negligible, and the costs considered in studies of system reconfiguration are those associated with losses, load shedding, etc.

However, the following objective function may be defined for applications in which switching costs are relevant and must be minimized:

$$z = \min\left\{\sum_{km \in \Psi_{SW}^{ON}} c_{km}^{SWITCH} \cdot (1 - \sigma_{km}) + \sum_{km \in \Psi_{SW}^{OFF}} c_{km}^{SWITCH} \cdot \sigma_{km}\right\} \quad (40)$$

where:

- Ψ_{SW}^{ON} Set of switchable circuits that were originally active (i.e., switched-on) at the situation corresponding to the input data for the ACOPF;

- Ψ_{SW}^{OFF} Set of switchable circuits that were originally inactive (i.e., switched-off) at the situation corresponding to the input data for the ACOPT;
- Ψ_{SW} Set of all switchable circuits in the system, $\Psi_{SW} = \Psi_{SW}^{ON} \cup \Psi_{SW}^{OFF}$;
- c_{km}^{SWITCH} Cost of switching action (cost of changing the status of the switchable circuit) associated with circuit km ;
- σ_{km} Binary variable that represent the desired state of the switchable circuit km ($\sigma_{km} = 1$ indicates that it is desired that the circuit is active; $\sigma_{km} = 0$ indicates that it is desired that the circuit is inactive).

3 SELECTED TECHNIQUES FOR THE REFORMULATION OF NON-LINEAR, NON-CONVEX PROBLEMS AS MIXED-INTEGER LINEAR PROGRAMS

The ACOPF for distribution system operations and expansion planning applications described in chapter 2 is a non-convex, mixed-integer non-linear programming (MINLP) problem. Problems of this class are usually difficult to treat computationally, and even the continuous relaxation of a non-convex MINLP is a global optimization problem [74], likely to be *NP*-hard (non-deterministic polynomial-time hard) [75].

There are, however, techniques that may be applied to approximate the nonlinearities of a MINLP, some of which may be employed to achieve approximations of arbitrary accuracy (i.e., with a level of accuracy arbitrated by the user), and reformulate the problem as a MILP. Solution algorithms for MILP, which are standard features in a wide range of commercially available solvers, may then be used to implicitly treat non-convexities, in a process that involves successively partitioning the domain of decisions variables.

The main advantages of reformulating MINLP problems as MILPs are well summarized by Geißler [61], in the excerpt reproduced below:

“The advantage of applying mixed integer linear techniques are that these methods are nowadays very mature, that is, they are fast, robust, and are able to solve problems with up to millions of variables. In addition, these methods have the potential of finding globally optimal solutions or at least to provide solution guarantees.”

As the excerpt indicates, one practical advantage of reformulating MINLPs as MILPs refers to the maturity of techniques for solving the latter class of problems. It is worth emphasizing that such maturity brings about not the only benefits with respect to computational performance listed in the excerpt, but also advantages associated with the availability of *commercial* solvers for mixed-integer linear programs. That is to say,

there are a number of companies that offer commercial-grade solvers that pertain to well-established product lines and may be used to solve mixed-integer linear programs. This is an important advantage from the point of view of industry applications, as it essentially translates into guarantees of longevity, maintainability and prevention of obsolescence of the solver that underlies an optimization solution.

The excerpt from reference [61] also mentions another class of benefits from employing MILP reformulations of MINLPs: the existence of solution guarantees. This also relates to another practical advantage associated with the use of commercial solvers: as those solvers usually provide the user with detailed execution reports, including information on the duality gap displayed on-screen in execution time, the user is able control the quality of the solutions obtained in the course of the algorithm execution and may, if desired, interrupt the optimization algorithm, accepting an intermediate solution *for which the value of the duality gap is known* (i.e., the *quality* of the solution can be controlled).

In the following sections, three convexification and linearization techniques for the reformulation of MINLPs as MILPs are presented. While presenting the techniques, some emphasis will be given to how the determination of the parameters for writing down the equality and inequality constraints may affect the accuracy of the approximation and the computational efforts associated with the solution of mixed-integer linear programs – a concept that will be loosely referred to as *tightness* in this dissertation.

The nomenclature used in this section applies exclusively to the presentation of the linearization and convexification techniques. None of the symbols used here should be interpreted as referring to any of the physical or economic quantities of the ACOF formulation (either the non-linear version presented at chapter 2 or the MILP reformulation presented at chapter 4).

3.1 Disjunctive constraints

In optimization problems involving binary decisions (i.e., decisions of the type *do/don't*), it may be required to represent *disjunctions* of the feasible region that are associated with values of binary decision variables [76]. A disjunction appears when,

according to the value of an auxiliary binary variable (a control variable), one set of constraints is enforced while another is relaxed.

For instance, assume that, in a given problem, either the constraint $\sum_j a_j^0 \cdot x_j \leq b^0$ is to be enforced when the binary variable d assumes the value $d = 0$, or the constraint $\sum_j a_j^1 \cdot x_j \leq b^1$ is to be enforced if $d = 1$. The enforcement of the former constraint implicates in the relaxation of the latter, and vice-versa. This *disjunction* may be modeled with help of the following *disjunctive constraints*:

$$\sum_j a_j^0 \cdot x_j - b^0 \leq M^0 \cdot d \quad (41)$$

$$\sum_j a_j^1 \cdot x_j - b^1 \leq M^1 \cdot (1 - d) \quad (42)$$

where the numerical value of the constants M^0 and M^1 must be large enough to ensure that constraint (41) is *relaxed* if $d = 1$ (i.e., that $\sum_j a_j^0 \cdot x_j - b^0$ will always be smaller than or equal to M^0), and that constraint (42) is relaxed if $d = 0$.

If each decision variable x_j is known to vary only within the interval $\underline{x}_j \leq x_j \leq \bar{x}_j$, the minimum value of the constants M^0 and M^1 that ensures that the desired constraints are relaxed can be pre-calculated by:

$$M^0 = \max\{\sum_j a_j^0 \cdot x_j - b^0\} \quad \text{subject to} \quad \underline{x}_j \leq x_j \leq \bar{x}_j \quad \forall j \quad (43)$$

$$M^1 = \max\{\sum_j a_j^1 \cdot x_j - b^1\} \quad \text{subject to} \quad \underline{x}_j \leq x_j \leq \bar{x}_j \quad \forall j \quad (44)$$

Disjunctive constraints may also be employed when more than two *disjunctions* of the feasible region need to be modeled.

For instance, consider the case in which only one of N constraints the type $\sum_j a_j^i \cdot x_j \leq b^i$, with $i \in \{1 \dots N\}$, is to be enforced at a time. A possible approach is to define N binary control variables d^i , with $i \in \{1 \dots N\}$, and to write the following set of equations:

$$\sum_j a_j^i \cdot x_j - b^i \leq M^i \cdot (1 - d^i) \quad (45)$$

$$\sum_i d^i = 1 \quad (46)$$

If, analogously to equations (43) and (44), if each decision variable x_j is known to vary only within the interval $\underline{x}_j \leq x_j \leq \bar{x}_j$, the minimum admissible value of each constant M^i can be pre-calculated by:

$$M^i = \max\{\sum_j a_j^i \cdot x_j - b^i\} \quad \text{subject to} \quad \underline{x}_j \leq x_j \leq \bar{x}_j \quad \forall j \quad (47)$$

The determination of the constants M^i , which are sometimes called *big-M* parameters and which will be referred to *disjunctive constants*¹⁰ in this dissertation, may be more complex than suggested by the explanation above – there are problems in which the solution of equation (47) is complex, the determination of the bounds of the interval $\underline{x}_j \leq x_j \leq \bar{x}_j$ is not immediate, or for which the other constraints of the problem may implicitly determine the actual range within which the decision variables may vary (which may be narrower than that defined simply by bounds informed as input parameters).

Nonetheless, defining disjunctive constants with the lowest possible absolute value is desired from the point of view of computational efficiency. This definition of *tight* values for the disjunctive constants is important because solution algorithms for MILPs include an intermediary *relaxation* step, in which integer decision variables are allowed to assume any continuous value – i.e., the associated integrality constraint is *relaxed*. Generally speaking, the closer the feasible space of this relaxed problem is to the *convex hull* of the original mixed-integer linear problem [76], the more computationally efficient will be the solution of a given mixed-integer linear program. The values of disjunctive constraints affect the size of the feasible space for the linear relaxations of the MILP [77]: defining *tight* disjunctive constraints will result in *tighter* linear relaxations – i.e., linear relaxations that are more tightly wrapped around the *convex hull*.

A few words on the nature of the procedure describe above are in order before moving on to section 3.2. The MILP problem obtained after applying disjunctive constraints to represent disjunctions of the feasible space is *not* convex. In fact, the phenomenon that we wish to represent, the disjunction of the feasible space,

¹⁰ We employ the term *disjunctive constant* in order not to necessarily associate the constants with the letter “M”, as other letters, besides “M”, will be also used to denote disjunctive constants in chapter 4.

corresponds to a non-convexity. However, the MILP formulation obtained by applying disjunctive constraints is treatable by solution algorithms that inherently treat non-convexities, by constructing a number of partitions of the feasible region and successively investigating these partitions, in an ordered fashion. Thus, disjunctive constraints are not a convexification technique *per se*, but a reformulation technique that allows treating non-convexities through algorithms designed to solve MILP problems.

3.2 Special ordered sets of type 2

The concept of ordered sets of decision variables, introduced in [78], may be used in two main classes of optimization applications:

- (i) Special ordered sets of type 1 (SOS1) are those in which no more than one variable may assume a non-zero value in the final solution of an optimization problem. Those ordered sets may be used to treat discrete functions that represent “*multiple choice problems*”, in which a single choice must be made among several discrete alternatives.
- (ii) Special ordered sets of type 2 (SOS2) are those in which no more than two variables may assume a non-zero value in the final solution of an optimization problem, and if two variables are non-zero they must be adjacent (consecutive in their ordering). SOS2 may be used to construct piecewise-linear approximations of non-convex, non-linear functions, such that these approximations can be integrated into a mixed-integer linear program.

The focus of this section will be on SOS2 and, particularly, on their application in piecewise-linear approximations of non-convex, non-linear functions.

Consider the example of Figure 3.1, which depicts a non-convex, non-linear function of a single variable, $f_{NL}(x)$, as well as its piecewise-linear approximation, $f(x)$. As indicated in the figure, the value of the function $f_{NL}(x)$ is calculated at N different *evaluation points* \hat{x}^i , with $i \in \{1 \cdots N\}$, thus resulting in N *evaluated values* \hat{f}^i . Linear segments are then obtained by constructing affine combinations of consecutive \hat{f}^i . Such an approximation will be linear within each segment, and therefore treatable through classical MILP techniques. The reader will notice that it is necessary that the affine combinations are constructed strictly with basis on consecutive evaluated values

\hat{f}^i , in order to preserve the representativeness of the approximated function with respect to the original, non-linear one – e.g., it is clear that an affine combination of the evaluated values \hat{f}^1 and \hat{f}^5 of Figure 3.1 would result in a linear segment that bears no resemblance with the original function.

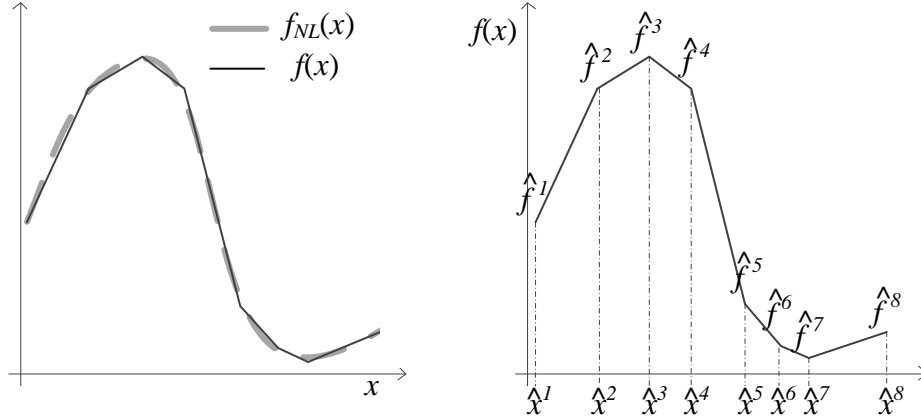


Figure 3.1: Piecewise-linearization $f(x)$ of non-linear functions $f_{NL}(x)$ and special ordered sets of type 2.

In order to ensure that the segments of the piecewise-linear approximation are built strictly with basis on affine combinations of *consecutive* \hat{f}^i , the weights associated with every evaluated value are treated as elements of an ordered set, and constraints are added to the MILP formulation to guarantee that at most two of the weights will assume a non-zero value in the final solution of the optimization problem, and any two non-zero values must be consecutive – i.e., the weights are treated as a SOS2. Naturally, as we are dealing with affine combinations, the weights must sum up to unity.

It is also necessary to obtain the argument of the approximated function that corresponds to the function value obtained by the affine combination of \hat{f}^i . The argument of the approximated function is obtained via an affine combination of the *evaluation points* \hat{x}^i , using the same weights employed for the affine combination of the *evaluated values* \hat{f}^i .

In the following, the mathematical formulation corresponding to the procedure described above is presented. Equation (48) corresponds to the *reference row* [78] of this formulation – the constraint by which the value of the argument x is obtained via the convex combination of the evaluated points \hat{x}^i :

$$\sum_{i=1}^N \lambda^i \cdot \hat{x}^i = x \tag{48}$$

In equation (87), the continuous decision variables λ^i , with $i \in \{1 \dots N\}$, are the weights for the affine combination, and pertain to a SOS2. The constraints for enforcing the special structure of this ordered set will be presented further in this section.

The approximated value of the function, f , is obtained with help of constraint (49), which is usually referred to as the *function row*:

$$\sum_{i=1}^N \lambda^i \cdot \hat{f}^i = f \quad (49)$$

The following constraint ensures that the weights λ^i sum up to unity, and is referred to as the *convexity row*:

$$\sum_{i=1}^N \lambda^i = 1 \quad (50)$$

It is now necessary to define constraints to ensure that the set of weights λ^i , with $i \in \{1 \dots N\}$, form a special ordered set of type 2 – i.e., constraints that impose that no more than two of those weights may assume non-zero values, and if two weights are non-zero they must be adjacent. This may be done by introducing N binary decision variables d^i , one for each weight λ^i , and defining the following constraints:

$$\sum_{i=1}^N d^i = 1 \quad (51)$$

$$\lambda^1 \leq d^1 \quad (52)$$

$$\lambda^i \leq d^{i-1} + d^i, \forall i \in \{2 \dots N\} \quad (53)$$

It is worth mentioning that specialized, efficient branching rules have been proposed for the solution of MILP with SOS2 constraints [78]. These specialized branching rules for SOS2 constraints are currently standard features in most commercial grade optimization solvers [75], [79], for the one-dimensional case.

The procedure presented above applies to the approximation of non-convex, non-linear functions of a single variable. This procedure can be extended for functions of higher dimension.

If a piecewise-approximation of a non-convex, non-linear function of two variables, $f_{NL}(x, y)$, is to be constructed, a possible alternative is to determine a *grid* of

evaluation points (\hat{x}^i, \hat{y}^j) , with $i \in \{1 \dots N^x\}$ and $j \in \{1 \dots N^y\}$, and determine the evaluated values $\hat{f}^{i,j} = f_{NL}(\hat{x}^i, \hat{y}^j)$ at each point of the grid. Within each region delimited by four vertices of the grid (the reader will notice that a *rectangular grid* is assumed here), the approximation f of the non-linear function will be obtained via an affine combination of the corresponding evaluated values. The set of weights $\lambda^{i,j}$ associated with each point (\hat{x}^i, \hat{y}^j) is ordered, and constraints must be added to ensure that no more than four weights may assume non-zero values, and that the weights that assume non-zero values are adjacent. A possible mathematical formulation for this procedure is presented below:

$$\sum_{i=1}^{N^x} \sum_{j=1}^{N^y} \lambda^{i,j} \cdot \begin{bmatrix} \hat{x}^i \\ \hat{y}^j \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad (54)$$

$$\sum_{i=1}^{N^x} \sum_{j=1}^{N^y} \lambda^{i,j} \cdot \hat{f}^{i,j} = f \quad (55)$$

$$\sum_{i=1}^{N^x} \sum_{j=1}^{N^y} \lambda^{i,j} = 1 \quad (56)$$

The following constraints impose the required structure on the set of weights $\lambda^{i,j}$:

$$\sum_{i=1}^{N^x} d^{x,i} = 1 \quad (57)$$

$$\lambda^{1,j} \leq d^{x,1} \quad , \forall j \in \{1 \dots N^y\} \quad (58)$$

$$\lambda^{i,j} \leq d^{x,i-1} + d^{x,i} \quad , \forall i \in \{2 \dots N^x\}, \forall j \in \{1 \dots N^y\} \quad (59)$$

$$\sum_{j=1}^{N^y} d^{y,j} = 1 \quad (60)$$

$$\lambda^{i,1} \leq d^{y,1} \quad , \forall i \in \{1 \dots N^x\} \quad (61)$$

$$\lambda^{i,j} \leq d^{y,j-1} + d^{y,j} \quad , \forall i \in \{1 \dots N^x\}, \forall j \in \{2 \dots N^y\} \quad (62)$$

where $d^{x,i}$, with $i \in \{1 \dots N^x\}$, and $d^{x,j}$, with $j \in \{1 \dots N^y\}$, are binary decision variables.

As previously stated, the procedure described above involves the construction of a *rectangular grid* of points at which the value of the non-linear function is evaluated. Procedures for constructing piecewise-linear approximations of non-linear functions of two variables based on constructing *triangular grids* of evaluation points (triangulation) have been proposed in the technical literature [61], [80], there being evidence that their

computational performance is superior to that of procedures based on *rectangular grids*. Yet, such procedures are not considered in this dissertation, and their application to the ACOPF in distribution systems will be the object of future work.

At this point, a few words on the computational requirements for the piecewise-linear approximation of non-convex, non-linear functions with SOS2 are in order. The computational requirements for these approximations grow significantly fast with the dimensions of the functions to be approximated [61]. Thus, non-convex functions of three decision variables are significantly more complex to treat than functions of two variables, and so on. It is worth mentioning that, as will be seen in chapter 4, the proposed formulation of the ACOPF requires only that functions of two arguments are approximated.

Another observation, immediately drawn from the equations presented in this section, is that the SOS2 approach to dealing with non-convex, non-linear functions of decision variables involves a trade-off between the desired level of approximation accuracy and the computational performance. For instance, Figure 3.1 clearly illustrates that, in arbitrating the number and location of *evaluation points* \hat{x}^l , the user can control the approximation accuracy. Nevertheless, using more evaluation points leads not only to increased accuracy, but also to an increased number of integer variables and constraints, which may lead to increased computational requirements. Obviously, the optimal trade-off between accuracy and computational performance depends on how severe the non-linearities of the function being approximated are. The results displayed in chapter 5 suggest that, for the ACOPF proposed in this dissertation, the computational requirements necessary to ensure satisfactorily accurate solutions are manageable.

Naturally, the choice of the points at which the non-linear function is evaluated directly affects both the accuracy of the piecewise-linear approximation and the computational requirements for the solution of the MILP. Again referring to the example of Figure 3.1, it is clear that adding an evaluation point between \hat{x}^4 and \hat{x}^5 would increase the number of integer variables without substantially increasing the quality of the approximation, and that removing the evaluation point \hat{x}^3 would significantly impact the accuracy of the approximation, despite of removing one integer variable. Thus, the *tightness* of the formulation is directly affected by the choice of evaluation points. The reader will notice that the term *tightness* is used here with a

slightly different meaning of that of section 3.1, but still in reference to the definition of parameters that affect accuracy and computational performance.

Before moving on, it is important to mention that using a smaller number of evaluation points will not *necessarily* lead to a faster solution of the MILP for all applications – e.g., if branch-and-bound is being used for the solution of the MILP, it may be that having more evaluation points leads to a particular pattern of investigation of the branch-and-bound tree that allows a faster convergence of the duality gap to zero.

3.3 Convex envelopes for bilinear products

Bilinear products are products of two continuous decision variables, such as $x \cdot y$. Those products are obviously non-convex and non-linear. A possible approach to treat bilinear products within linear programming (LP) formulations (and by extension within MILP formulations) is to substitute them by an auxiliary variable, z , and then define constraints that are linear functions of x and y and that bound z within a narrow interval around the true value of $x \cdot y$.

The most general case of this approach is to define linear constraints that bound z *from below* and *from above* – respectively, a *convex underestimator* and a *concave overestimator* for the bilinear product. A convex under-estimator is a function $u(x, y)$ such that $u(x, y) \leq x \cdot y$ for all values that x and y may assume. Analogously, a concave over-estimator is a function $o(x, y)$ such that $o(x, y) \geq x \cdot y$ in the domain of interest. Together, these form the *convex envelope* for the bilinear product.

The definition of the last paragraph correctly suggests that many different functions may serve as *convex underestimators* and *concave overestimators*. However, there is obviously interest in defining the tightest possible *convex envelope* for the bilinear product. As the auxiliary variable z will be allowed to assume any value in the interval $u(x, y) \leq z \leq o(x, y)$, the maximum potential approximation error will obviously depend on how *tightly* the envelope wraps the bilinear product – i.e., on how significant the differences $x \cdot y - u(x, y)$ and $o(x, y) - x \cdot y$ can be.

The tightest possible convex envelope for bilinear products $x \cdot y$ has been determined by McCormick [81], and is thus commonly referred to as *McCormick's envelope*. Assuming x bounded within the interval $\underline{x} \leq x \leq \bar{x}$ and y bounded within

$\underline{y} \leq y \leq \bar{y}$, McCormick's envelope for the auxiliary variable z is defined with help of the following linear constraints:

$$z \geq \underline{x} \cdot y + x \cdot \underline{y} - \underline{x} \cdot \underline{y} \quad (63)$$

$$z \geq \bar{x} \cdot y + x \cdot \bar{y} - \bar{x} \cdot \bar{y} \quad (64)$$

$$z \leq \underline{x} \cdot y + x \cdot \bar{y} - \underline{x} \cdot \bar{y} \quad (65)$$

$$z \leq \bar{x} \cdot y + x \cdot \underline{y} - \bar{x} \cdot \underline{y} \quad (66)$$

where equations (63) and (64) correspond to the convex under-estimator and equations (65) and (66) to the concave over-estimator for $x \cdot y$.

As previously stated, McCormick's envelope is the tightest possible convex envelope for bilinear products. The tightness of McCormick's envelope *for each application*, however, depends on how accurate the upper and lower bounds of the intervals $\underline{x} \leq x \leq \bar{x}$ and $\underline{y} \leq y \leq \bar{y}$ are defined.

In order to understand that, consider that $\underline{x} = \underline{y} = 1.0$ and $\bar{x} = \bar{y} = 1.5$ for a certain application. Assume, however, that a mistake was inadvertently made while defining the upper and lower bounds for the variation of x and y , and the lower bounds for the interval were wrongfully taken as $\underline{x} = \underline{y} = 0.5$. Figure 3.2 indicates the actual value of the product $x \cdot y$, as well as the convex underestimator and the concave overestimator for McCormick's envelope when both x and y are incorrectly considered to vary within $[0.5, 1.5]$. In this case, the absolute value of the approximation error within the correct domain (i.e., $[1.0, 1.5]$ for x and y) may be as high as 0.25.

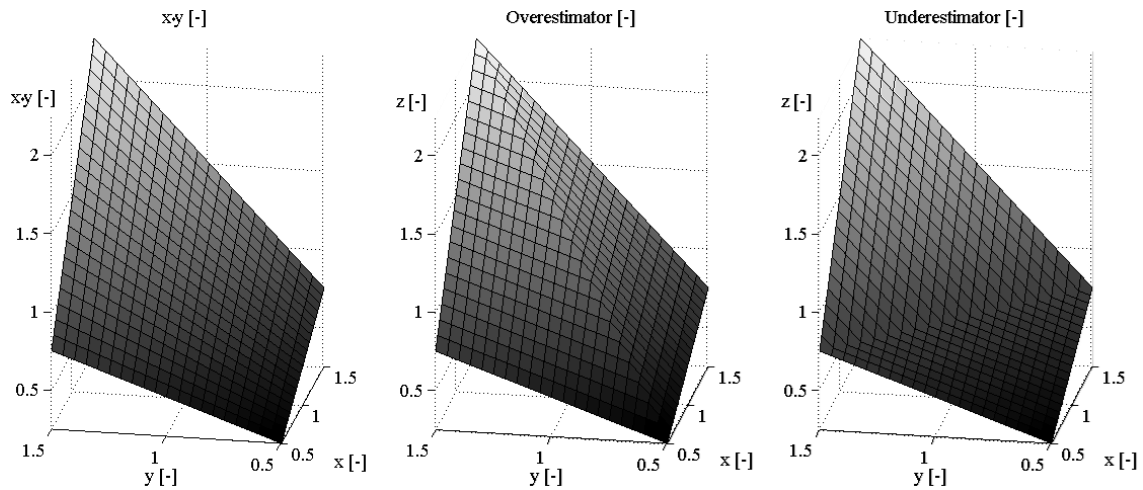


Figure 3.2: Bilinear products and *McCormick's envelope*, considering the *incorrect* lower bound of 0.5 for x and y : $z = x \cdot y$ (left); overestimator for z (middle); underestimator for z (right).

Consider now that the mistake has been identified and corrected, and that the correct lower bound $\underline{x} = \underline{y} = 1.0$ has been considered while constructing McCormick's envelope for the bilinear product. In this situation, the convex underestimator and the concave overestimator indicated in Figure 3.3 would be obtained. In order to facilitate the comparison, the range of the axes of all graphs in Figure 3.3 matches that of Figure 3.2. After a comparison of the figures, it becomes clear that the convex envelope indicated in Figure 3.3 is much *tighter* within the domain of interest – i.e., for x and y varying within $[1.0, 1.5]$ – than that of Figure 3.2. In fact, now the maximum absolute value of the approximation error within the domain of interest is 0.0625 (a significant decrease over the 0.25 of the previous paragraph).

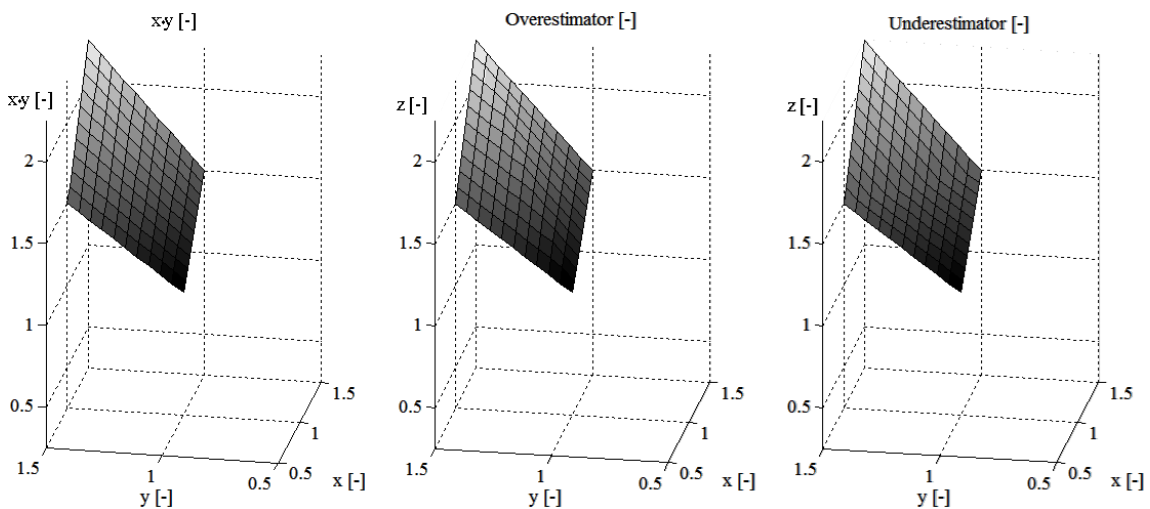


Figure 3.3: Bilinear products and *McCormick's envelope*, consider the *correct* lower bound of 1.0 for x and y : $z = x \cdot y$ (left); overestimator for z (middle); underestimator for z (right).

The previous analysis illustrates the fact that, for any given application, the accuracy of the approximation obtained by McCormick's envelope will be dictated by how *tight* one is able to define the upper and lower bounds on the values that the continuous variables may assume. The closer these parameters match the actual interval in which the continuous decision variables may vary, the better the approximation will be.

Before moving on to the next chapter, it is worth briefly mentioning that analytical expressions for convex envelopes for trilinear and quadrilinear terms (respectively, products of three and four continuous decisions variables) have been proposed in the technical literature [82], [83]. Those are not employed in the MILP formulation proposed in this dissertation.

4 THE MILP REFORMULATION OF THE ACOPF FOR DISTRIBUTION SYSTEMS

This chapter presents the main technical contribution of this dissertation: the MILP reformulation of the ACOPF problem for distribution systems, with focus on operations and expansion planning applications.

In Section 4.1, the main characteristics of the proposed MILP reformulation of the ACOPF for distribution systems are presented, and the practical advantages associated with these characteristics are discussed in detail. The mathematical formulation of the mixed-integer linear program is presented in section 4.2.

While defining the mathematical expressions of section 4.2, reference will be made to a number of parameters that are needed for the use of the linearization and convexification techniques defined in chapter 3: (i) the *disjunctive constants* necessary for the definition of *disjunctive constraints*; (ii) the *evaluation points* and *evaluated values* necessary for the definition of *piecewise-linear approximations with SOS2*; and (iii) the *upper and lower bounds* for the continuous variables whose product is modeled with help of *McCormick's envelope*. As seen in chapter 3, the definition of these parameters affects the accuracy of the approximations and/or the computational requirements for the solution of the resulting mixed-integer linear program. Section 4.3 will deal with the definition of these parameters, taking advantage of particular characteristics of the distribution system in order to achieve satisfactory trade-offs between accuracy and computational performance.

In section 4.4, reference is made to an alternative MILP reformulation of the ACOPF in distribution systems. This alternative formulation, which is thoroughly presented in Appendix B (chapter 8), has been investigated as part of the research activities that led to the present dissertation, but abandoned at early stages due to its performance being inferior, with respect to accuracy and computational requirements, to the formulation presented in this fourth chapter.

4.1 Main characteristics of the proposed formulation

The main characteristics of the proposed MILP reformulation of the ACOPF for distribution systems are directly related to the advantageous features of the proposed formulation, which have been already mentioned in the introductory section of this dissertation, and are reproduced below for the sake of clarity:

- (i) The proposed formulation captures the non-linear behavior of the distribution system with an arbitrarily accurate approximation.
- (ii) The proposed formulation supports both continuous and discrete decisions, respectively via continuous and integer decision variables.
- (iii) The proposed formulation is constructed with basis on conventional physical variables that describe network behavior (bus voltages, branch currents, bus power injections, etc.), yielding significant flexibility in defining a number of possible objective functions for the ACOPF, and extending its applicability to a number of different problems faced by distribution system engineers.
- (iv) The proposed formulation can be solved to global optimality with the use of widely employed and commercially available mixed-integer linear optimization solvers.

The direct relationship of the abovementioned features to the characteristics of the proposed formulation will become clear with the discussion of the next subsections.

4.1.1 Rectangular coordinates, current-voltage formulation of Kirchhoff's laws

The first relevant characteristic is that the proposed formulation is based on expressing complex variables in rectangular coordinates (real and imaginary components, as opposed to angles and magnitudes), and utilizing voltages and currents (as opposed to voltages and power quantities) to describe Kirchhoff's laws. As seen in section 2.2.1.1 of this document, this leads to the linearity of the set of constraints describing Kirchhoff's laws. Thus, these linear constraints can be immediately factored into a MILP problem, without the need to employ any transformation (e.g., linearization

or convexification) that may result in approximations or demand the use of integer variables.

As seen in chapter 2, employing rectangular coordinates and describing the behavior of the network via current-voltage equations brings about some non-linearities that would not be verified if the power-voltage formulation of network equations were used: obtaining the current injections corresponding to the power injections of *generators* and *loads of the constant-power and constant-current types* require the use of non-linear equations. Nonetheless, the linearization and convexification techniques described in chapter 3 allow dealing with these latter non-linearities efficiently – partially due to the fact that specific characteristics of the distribution system (particularly those described in section 2.1.2, which result in the voltage angles of all buses of typical distribution systems varying within narrow intervals around $\theta^{ref} = 0^\circ$) allow conciliating accuracy and computational performance, as we will see later in this chapter.

It is important to emphasize that, even if the power-voltage description of Kirchhoff's laws were to be used, representing loads of the *constant-current type* would demand non-linear equations. Besides, loads of the *constant-impedance type*, which are described exclusively via linear equations when Kirchhoff's laws are described with current-voltage quantities, would require non-linear equations for their description in case Kirchhoff's laws were formulated with basis on voltages and power quantities.

One last advantage of using the current-voltage description of Kirchhoff's laws is that this facilitates the formulation of constraints representing thermal loading limits of overhead lines. Such thermal loading limits are associated with *maximum admissible currents* (despite the fact that these limits are commonly approximated as limits on apparent power flows in many applications), and currents are “natural” decision variables in the proposed formulation.

4.1.2 Use of integer decision variables

As stated in the introductory chapter of this dissertation, there are a number of applications in distribution systems operations and expansion planning that involve discrete decisions. The most traditional of these relate to binary decisions on reinforcements to the network (either build *or* do not build the reinforcement) and to

network reconfiguration (a switch is either on *or* off). Also, depending on the level of detail of the representation of the network and on the necessity to accurately represent operation actions of the distribution company, other discrete decisions may need to be modeled – e.g., the reader will recall that most distribution utilities currently implement load shedding by de-energizing entire segments of the primary distribution feeder system. If this procedure is to be simulated, the decision to shed any given load at the distribution system is discrete, as the load is either energized *or* de-energized.

If classical mathematical programming techniques are to be used for the formulation and solution of the ACOPF, the representation of discrete decisions such as those mentioned above require the use of integer decision variables. Notably, binary decisions (of the type *do or don't*) may be formulated by using binary decision variables (which may only assume the values 0 or 1). Naturally, it is also required to represent continuous decisions in the ACOPF problem for distribution system, in order to allow answering questions such as *how much to import from an external network*, or *how much should the output of a given generator be*.

Also, the definition of certain type of constraints may require the use of integer variables. This is the case of constraints for ensuring network radiality, which will be defined further in this chapter.

4.1.3 Treatment of non-convexities and non-linearities

The need to model discrete decisions is not the only reason for employing integer decision variables in the proposed reformulation of the ACOPF problem. As seen in section 3.2, the piecewise-linear approximation of non-convex, non-linear functions based on using SOS2 also requires that binary variables are used, in order to impose a certain structure on *ordered sets* of decision variables. It is the structure imposed by binary constraints that allow defining the segments of the piecewise-linear approximation exclusively as convex combinations of *adjacent* evaluated values, and it is the binary variables that contain the information of *which segment* of the piecewise-linear approximated function is *active* at the solution of the optimization problem. The auxiliary binary decision variables are of uttermost importance: as only one of the linear segments of the piecewise-linear approximation is active at a time, and as a linear

segment is convex (and obviously linear) by definition, the piecewise-linear approximation may be factored into a mixed-integer linear program.

Also, the procedure described in section 3.2 may be used to obtain approximations of arbitrary accuracy – i.e., with a level of accuracy arbitrated by the user and directly related to the number of segments used to approximate the original non-convex, non-linear function. Enhanced accuracy comes at the cost of augmented computational complexity – but the case study results in chapter 5 will show that, for the problem at hand and when particular characteristics of the distribution system are correctly taken into account while determining the parameters used to write down the constraints, satisfactory compromises between accuracy and computational performance can be achieved. This is partially related to the fact that the approximated functions have low dimensions (i.e., they are not functions with a large number of arguments) and are fairly well behaved.

It is worth mentioning that, among the two techniques presented in chapter 3 for producing approximations of non-convex, non-linear functions, only that based on piecewise-linear approximations with the use of SOS2 constraints (section 3.2) may always have its accuracy directly controlled by the user. The accuracy of the approximation obtained with McCormick’s envelope (section 3.3) for products of two continuous decision variables $x \cdot y$ is implicitly determined by the lower and upper bounds on x and y . However, the reader will notice that there are no impediments for employing a SOS2-based piecewise-linear approximation of products of two continuous decision variables – thus, if needed, the MILP reformulation of the ACOPF may be made entirely independent of McCormick’s envelope, which results in the approximation accuracy always being controlled by the user. In fact, a SOS2-based reformulation of the bilinear products that appear in the constraints used for obtaining generator current injections is presented in Appendix C (section 9) of this dissertation, and used in the case study of section 5.2.3.

4.1.4 Final formulation as a MILP

All of the features mentioned above can be accommodated within a MILP formulation. This leads to a class of benefits that can hardly be overestimated, and are associated with the maturity of the techniques and commercial-grade software packages

dedicated to the solution of mixed-integer linear programs, as extensively described at the beginning of chapter 3.

4.2 Mathematical formulation

As in chapter 2, the constraints employed for modeling the behavior of the network and enforcing operating limits will be presented first. This is done in subsection 4.2.1. Objective functions for selected distribution system operations and expansion planning applications will be presented in subsection 4.2.2.

4.2.1 Constraints: modeling electrical behavior and enforcing operating limits

4.2.1.1 Kirchhoff's Laws

The constraints presented in subsection 2.2.1.1 for modeling Kirchhoff's Current Law are entirely linear, and can be factored into a MILP without any modification. For the sake of clarity, constraints (1) and (2) of subsection 2.2.1.1 are reproduced below:

$$I_{d,k}^{re} + \sum_{m \in \Omega_k} I_{km}^{re} = I_{g,k}^{re} + \sum_{m \in \Omega_k} I_{mk}^{re} \quad , \forall k \in \Omega_B \quad (67)$$

$$I_{d,k}^{im} + \sum_{m \in \Omega_k} I_{km}^{im} = I_{g,k}^{im} + \sum_{m \in \Omega_k} I_{mk}^{im} \quad , \forall k \in \Omega_B \quad (68)$$

One of the main reasons for proposing a MILP reformulation of the ACOPF was to model decisions regarding the change of status of branches of the network: branches may be *active* or *inactive*. For switchable circuits, the states *active* and *inactive* correspond to *switched-on* or *switched-off*; for candidate reinforcements for distribution system expansion, these states correspond to *built* or *not built*.

Normally, it is not all branches of the distribution network that can have their status modified: there may be many existing, non-switchable branches that are always active. For these circuits, the following constraints model Kirchhoff's Voltage Law:

$$V_k^{re} - V_m^{re} = I_{km}^{re} \cdot R_{km} - I_{km}^{im} \cdot X_{km} \quad , \forall km \in \{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\} \quad (69)$$

$$V_k^{im} - V_m^{im} = I_{km}^{re} \cdot X_{km} + I_{km}^{im} \cdot R_{km} \quad , \forall km \in \{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\} \quad (70)$$

The reader will notice that the expressions above are virtually identical to (3) and (4) of section 2.2.1.1 – the only difference being that (68) and (69) are not defined for all branches in the system (the set Ψ_C), but only for non-switchable and non-candidate branches (i.e., branches in the set $\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$).

For branches whose status corresponds to a decision variable of the ACOPF (switchable branches or candidate branches), it is necessary that Kirchhoff's Voltage Law is enforced whenever the circuit is active (i.e., whenever $\sigma_{km} = 1$), but relaxed whenever the branch is inactive (i.e., whenever $\sigma_{km} = 0$). In order to do that, the following disjunctive constraints are defined:

$$W_{km}^{VL,re,l} \cdot (1 - \sigma_{km}) \leq V_k^{re} - V_m^{re} - I_{km}^{re} \cdot R_{km} + I_{km}^{im} \cdot X_{km} \leq W_{km}^{VL,re,u} \cdot (1 - \sigma_{km})$$

$$, \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (71)$$

$$W_{km}^{VL,im,l} \cdot (1 - \sigma_{km}) \leq V_k^{im} - V_m^{im} - I_{km}^{re} \cdot X_{km} - I_{km}^{im} \cdot R_{km} \leq W_{km}^{VL,im,u} \cdot (1 - \sigma_{km})$$

$$, \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (72)$$

where:

$$W_{km}^{VL,re,l} ; W_{km}^{VL,re,u}$$

Disjunctive constants for Kirchhoff's Voltage Law (difference among the real components of terminal bus voltages);

$$W_{km}^{VL,im,l} ; W_{km}^{VL,im,u}$$

Disjunctive constants for Kirchhoff's Voltage Law (difference among the imaginary components of terminal bus voltages).

In section 4.3.1, it is shown how to determine the constants defined above.

Constraints (71) to (72) are not the only disjunctive constraints that need to be formulated to ensure the correct modeling of inactive branches. Obviously, the real and imaginary components of the current flowing through inactive branches must be forced to zero. In order to do that, the following disjunctive constraints are added to the MILP model:

$$-W_{km}^{CL} \cdot \sigma_{km} \leq I_{km}^{re} \leq W_{km}^{CL} \cdot \sigma_{km} \quad , \forall k \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (73)$$

$$-W_{km}^{CL} \cdot \sigma_{km} \leq I_{km}^{im} \leq W_{km}^{CL} \cdot \sigma_{km} \quad , \forall k \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (74)$$

where:

W_{km}^{CL} Disjunctive constant for the disjunctive constraints that force the real and imaginary parts of the current flowing through an inactive branch to zero.

The definition of W_{km}^{CL} will also be dealt with in section 4.3.1.

4.2.1.2 Generation

At this point, it is adequate to recall the definition of the following sets:

- Ω_{GEN} Set of all buses to which generators connect;
- Ω_{CTRPQ} Set of buses to which generators with control over the output of active and reactive power connect;
- Ω_{CTRQ} Set of buses to which generators with non-controllable active power output (but with reactive power output assumed to be controllable) connect;
- Ω_{CURT} Set of buses to which curtailable generators connect;
- Ω_{NCRT} Set of buses to which non-curtailable generators connect.

It is assumed that, for the distribution system planning applications of interest, there will be no need to associate costs with the curtailment of generators with controllable active power output. Therefore, only generators with non-controllable power output may be in Ω_{CURT} . – i.e., the intersection $\Omega_{CURT} \cap \Omega_{CTRPQ}$ corresponds to an empty set.

In the following subsections, the mathematical formulation of the constraints that model current injections for each type of generator is presented. The formulation of the following subsections makes use of McCormick’s envelopes. As previously stated, a formulation that eliminates the need to employ McCormick’s envelopes, and relies solely on SOS2-based piecewise-linear approximations, is presented in Appendix C (section 9) of this dissertation.

4.2.1.2.1 Non-curtailable generators with no control over the active power output

The current injections from generators that pertain to $\{\Omega_{CTRQ} \cap \Omega_{NCRT}\}$ are modeled with help of the following constraints:

$$I_{g,k}^{re} = \xi_k \cdot g_k^p + m_k^{Q,re} \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (75)$$

$$I_{g,k}^{im} = \zeta_k \cdot g_k^p - m_k^{Q,im} \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (76)$$

where:

$m_k^{Q,re}$ Auxiliary decision variable for modeling the product $\zeta_k \cdot g_k^Q$;

$m_k^{Q,im}$ Auxiliary decision variable for modeling the product $\xi_k \cdot g_k^Q$.

The auxiliary decision variables $m_k^{Q,re}$ and $m_k^{Q,im}$ are free in signal.

The reader will recall that, for generators with no control over the active power output, g_k^p is a parameter (and not a decision variable).

There are a number of constraints needed for defining the auxiliary decision variables that appear in equations (76) and (77): ξ_k , ζ_k , $m_k^{Q,re}$ and $m_k^{Q,im}$.

The auxiliary variables ξ_k and ζ_k will be approached first. As seen in section 2.2.1.2, these auxiliary decision variables represent non-convex, non-linear functions of V_k^{re} and V_k^{im} – i.e., they are both functions of two variables. For the MILP reformulation of the ACOPF, piecewise-linear approximations of these non-convex, non-linear functions will be employed. Using the technique based on SOS2 and described in section 3.2, the following set of equations may be used for the definition of ξ_k and ζ_k :

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \lambda_k^{r,s} \cdot \begin{bmatrix} \hat{\xi}_k^{r,s} \\ \hat{\zeta}_k^{r,s} \end{bmatrix} = \begin{bmatrix} \xi_k \\ \zeta_k \end{bmatrix} \quad , \forall k \in \{\Omega_{GEN} \cup \Omega_{PCTE}\} \quad (77)$$

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \lambda_k^{r,s} \cdot \begin{bmatrix} \hat{V}_k^{re,r} \\ \hat{V}_k^{im,s} \end{bmatrix} = \begin{bmatrix} V_k^{re} \\ V_k^{im} \end{bmatrix} \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (78)$$

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \lambda_k^{r,s} = 1 \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (79)$$

where:

Γ^{re}	Set of indices for <i>evaluation points</i> $\hat{V}_k^{re,r}$ and associated variables;
Γ^{im}	Set of indices for <i>evaluation points</i> $\hat{V}_k^{im,s}$ and associated variables;
$\hat{V}_k^{re,r}$	Evaluation points of real component of voltage at bus k ;
$\hat{V}_k^{im,s}$	Evaluation points of imaginary component of voltage at bus k ;
$\hat{\xi}_k^{r,s}$	Evaluated values of function $\xi_k(V_k^{re}, V_k^{im})$, for bus k ;
$\hat{\zeta}_k^{r,s}$	Evaluated values of function $\zeta_k(V_k^{re}, V_k^{im})$, for bus k ;
$\lambda_k^{r,s}$	Weights for constructing piecewise-linear approximation of non-convex, non-linear functions of V_k^{re} and V_k^{im} .

Section 4.3.2 deals with the definition of the *evaluation points* and *evaluated values* $\hat{V}_k^{re,r}$, $\hat{V}_k^{im,s}$, $\hat{\xi}_k^{r,s}$ and $\hat{\zeta}_k^{r,s}$.

The vector equations (77) and (78) correspond respectively to the *function row* and to the *reference row* for the piecewise-linear approximation, while (79) is the *convexity row*.

The reader will notice that equations (78) and (79) are defined for all buses in the system, except the voltage reference buses. This is due to the fact these same equations will be used for constructing a piecewise-linear approximation of the square root function through which the voltage magnitude of each bus is obtained, as described in subsection 4.2.1.4.1. The reader will recall that the voltage magnitude for the voltage reference bus is either fixed or it consists of a “natural” decision variable, and therefore the *implicit determination* of the bus voltage magnitude at voltage reference buses is not necessary.

Equation (77) is defined for all buses with generators and all buses with constant-power loads, as these are the buses for which the auxiliary variables ξ_k and ζ_k are defined, as these variables are needed to obtain the current injections corresponding to power injections.

Having defined (77) to (79), it is necessary to define constraints that ensure that the weights $\lambda_k^{r,s}$ form a SOS2:

$$\sum_{r \in \Gamma^{re}} x_k^r = 1, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (80)$$

$$\lambda_k^{1,s} \leq x_k^1, \forall s \in \Gamma^{im}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (81)$$

$$\lambda_k^{r,s} \leq x_k^{r-1} + x_k^r, \forall r \in \{\Gamma^{re} \setminus \{1\}\}, s \in \Gamma^{im}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (82)$$

$$\sum_{s \in \Gamma^{im}} y_k^s = 1, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (83)$$

$$\lambda_k^{r,1} \leq y_k^1, \forall r \in \Gamma^{re}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (84)$$

$$\lambda_k^{r,s} \leq y_k^{s-1} + y_k^s, \forall r \in \Gamma^{re}, s \in \{\Gamma^{im} \setminus \{1\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (85)$$

where:

$x_k^r ; y_k^s$ Auxiliary binary decision variables.

Having dealt with the definition of ξ_k and ζ_k , it is necessary to indicate the constraints for the definition of $m_k^{Q,re}$ and $m_k^{Q,im}$. As previously stated, these auxiliary variables are used for approximating the product of continuous decision variables. For their definition, it is possible either to use piecewise-linear approximations or to employ McCormick's envelope.

At this point, an option is made for the latter procedure, and the following constraints are defined:

$$m_k^{Q,re} \geq \underline{\zeta}_k \cdot g_k^Q + \zeta_k \cdot \underline{g}_k^Q - \underline{\zeta}_k \cdot \underline{g}_k^Q, \forall k \in \Omega_{GEN} \quad (86)$$

$$m_k^{Q,re} \geq \bar{\zeta}_k \cdot g_k^Q + \zeta_k \cdot \bar{g}_k^Q - \bar{\zeta}_k \cdot \bar{g}_k^Q, \forall k \in \Omega_{GEN} \quad (87)$$

$$m_k^{Q,re} \leq \underline{\zeta}_k \cdot g_k^Q + \zeta_k \cdot \bar{g}_k^Q - \underline{\zeta}_k \cdot \bar{g}_k^Q, \forall k \in \Omega_{GEN} \quad (88)$$

$$m_k^{Q,re} \leq \bar{\zeta}_k \cdot g_k^Q + \zeta_k \cdot \underline{g}_k^Q - \bar{\zeta}_k \cdot \underline{g}_k^Q, \forall k \in \Omega_{GEN} \quad (89)$$

$$m_k^{Q,im} \geq \underline{\xi}_k \cdot g_k^Q + \xi_k \cdot \underline{g}_k^Q - \underline{\xi}_k \cdot \underline{g}_k^Q, \forall k \in \Omega_{GEN} \quad (90)$$

$$m_k^{Q,im} \geq \bar{\xi}_k \cdot g_k^Q + \xi_k \cdot \bar{g}_k^Q - \bar{\xi}_k \cdot \bar{g}_k^Q, \forall k \in \Omega_{GEN} \quad (91)$$

$$m_k^{Q,im} \leq \underline{\xi}_k \cdot g_k^Q + \xi_k \cdot \bar{g}_k^Q - \underline{\xi}_k \cdot \bar{g}_k^Q, \forall k \in \Omega_{GEN} \quad (92)$$

$$m_k^{Q,im} \leq \bar{\xi}_k \cdot g_k^Q + \xi_k \cdot \underline{g}_k^Q - \bar{\xi}_k \cdot \underline{g}_k^Q, \forall k \in \Omega_{GEN} \quad (93)$$

where:

$\underline{\xi}_k ; \bar{\xi}_k$ Lower and upper bounds for the values that ξ_k may assume;

$\underline{\zeta}_k ; \bar{\zeta}_k$ Lower and upper bounds for the values that ζ_k may assume.

The lower and upper bounds for g_k^Q are inputs for the ACOPF, as they depend on particular characteristics of each generator. The definition of the lower and upper bounds for ξ_k and ζ_k will be dealt with in section 4.3.3.

The reader will notice that constraints (86) to (93) have been defined for all generators of the system (i.e., $\forall k \in \Omega_{GEN}$), and not only for the generators that pertain to $\{\Omega_{CTRPQ} \cap \Omega_{NCRT}\}$. This is due to the fact that all generators in the system are assumed to have control over their reactive power output, and it is therefore necessary to determine the approximation of the bilinear products $\zeta_k \cdot g_k^Q$ and $\xi_k \cdot g_k^Q$ for the whole set Ω_{GEN} .

It is worth mentioning that, despite the fact that McCormick's envelopes have been used for the formulation of the constraints used for obtaining the current injections from generators in in this section, there are alternative formulations that completely eliminate the need to employ the convex envelopes. An alternative formulation, based on treating the generator currents $I_{g,k}^{re}$ and $I_{g,k}^{im}$ as functions of three continuous decision variables – i.e., $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q)$ and $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q)$ – and constructing a piecewise-linear approximation of these functions with help of SOS2, is presented in Appendix C (section 9.1) of this dissertation.

4.2.1.2.2 Curtailable generators with no control over the active power output

As already discussed in subsection 2.2.2.2, generation curtailment is considered to be a discrete decision in the proposed formulation: the generator at bus k will be considered to be either energized ($\tau_k = 0$) or de-energized ($\tau_k = 1$).

Therefore, it is necessary to ensure that, if the generator connected to bus k is curtailed, its current injections will be forcefully set to zero. In order to do that, the following set of *disjunctive constraints* will be defined for generators that pertain to $\{\Omega_{CTRPQ} \cap \Omega_{CURT}\}$ (i.e., for curtailable generators with no control over their active power input):

$$M_k^{G,re,1} \cdot \tau_k \leq I_{g,k}^{re} - \xi_k \cdot g_k^P - m_k^{Q,re} \leq M_k^{G,re,2} \cdot \tau_k, \quad \forall k \in \{\Omega_{CTRPQ} \cap \Omega_{CURT}\} \quad (94)$$

$$M_k^{G,re,3} \cdot (1 - \tau_k) \leq I_{g,k}^{re} \leq M_k^{G,re,4} \cdot (1 - \tau_k) \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (95)$$

$$M_k^{G,im,1} \cdot \tau_k \leq I_{g,k}^{im} - \zeta_k \cdot g_k^P + m_k^{Q,im} \leq M_k^{G,im,2} \cdot \tau_k \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (96)$$

$$M_k^{G,im,3} \cdot (1 - \tau_k) \leq I_{g,k}^{im} \leq M_k^{G,im,4} \cdot (1 - \tau_k) \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (97)$$

where:

$$M_k^{G,re,1} ; M_k^{G,re,2} ; M_k^{G,re,3} ; M_k^{G,re,4} ; M_k^{G,im,1} ; M_k^{G,im,2} ; M_k^{G,im,3} ; M_k^{G,im,4}$$

Disjunctive constants for the disjunctive constraints employed for modeling generation curtailment.

The definition of these disjunctive constants will be dealt with in section 4.3.1.

The reader will notice that the constraints needed for determining the value all auxiliary decision variables that appear in (94) to (97) have already been defined, as many of the constraints of previous sections have been defined for sets that include $\{\Omega_{CTRQ} \cap \Omega_{CURT}\}$ as a subset.

4.2.1.2.3 Generators with control over the active power output

Generators with control over the active power output are considered to be non-curtailable. This assumption is based on the fact that, as these generators can simply set their output to zero, it is not required to model their curtailment and to attribute a cost to it.

The current injections from generators that pertain to Ω_{CTRPQ} are modeled with help of the following constraints:

$$I_{g,k}^{re} = m_k^{P,re} + m_k^{Q,re} \quad , \forall k \in \Omega_{CTRPQ} \quad (98)$$

$$I_{g,k}^{im} = m_k^{P,im} - m_k^{Q,im} \quad , \forall k \in \Omega_{CTRPQ} \quad (99)$$

where:

$$m_k^{P,re} \quad \text{Auxiliary decision variable for modeling the product } \xi_k \cdot g_k^P;$$

$m_k^{P,im}$ Auxiliary decision variable for modeling the product $\zeta_k \cdot g_k^P$.

For the generators that pertain to Ω_{CTRPQ} , g_k^P is a continuous decision variable. Thus, products of two decision variables appear once again. These products may be approximated either by a piecewise-linear function constructed with help of SOS2, or via McCormick's envelope. An option is made for the latter procedure, and the following constraints are defined:

$$m_k^{P,re} \geq \underline{\xi}_k \cdot g_k^P + \xi_k \cdot \underline{g}_k^P - \underline{\xi}_k \cdot \underline{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (100)$$

$$m_k^{P,re} \geq \bar{\xi}_k \cdot g_k^P + \xi_k \cdot \bar{g}_k^P - \bar{\xi}_k \cdot \bar{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (101)$$

$$m_k^{P,re} \leq \underline{\xi}_k \cdot g_k^P + \xi_k \cdot \bar{g}_k^P - \underline{\xi}_k \cdot \bar{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (102)$$

$$m_k^{P,re} \leq \bar{\xi}_k \cdot g_k^P + \xi_k \cdot \underline{g}_k^P - \bar{\xi}_k \cdot \underline{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (103)$$

$$m_k^{P,im} \geq \underline{\zeta}_k \cdot g_k^P + \zeta_k \cdot \underline{g}_k^P - \underline{\zeta}_k \cdot \underline{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (104)$$

$$m_k^{P,im} \geq \bar{\zeta}_k \cdot g_k^P + \zeta_k \cdot \bar{g}_k^P - \bar{\zeta}_k \cdot \bar{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (105)$$

$$m_k^{P,im} \leq \underline{\zeta}_k \cdot g_k^P + \zeta_k \cdot \bar{g}_k^P - \underline{\zeta}_k \cdot \bar{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (106)$$

$$m_k^{P,im} \leq \bar{\zeta}_k \cdot g_k^P + \zeta_k \cdot \underline{g}_k^P - \bar{\zeta}_k \cdot \underline{g}_k^P, \forall k \in \Omega_{CTRPQ} \quad (107)$$

The auxiliary decision variable $m_k^{P,re}$ may be free in sign or non-negative, depending on the upper and lower bounds defined for g_k^P , whereas $m_k^{P,im}$ is always free in sign. Typically, g_k^P and $m_k^{P,re}$ will be non-negative. The lower and upper bounds for g_k^P are inputs for the ACOPT, and vary by generator. The constraints needed for defining all auxiliary decision variables that appear in (100) to (107) have already been defined.

A discussion similar to that of the end of subsection 4.2.1.2.1 applies here: it is possible to define an alternative formulation of the constraints used for obtaining the current injections from generators that control their active power output that completely eliminates the need to employ McCormick's envelopes. This formulation is based on treating the generator currents as functions of four decision variables – i.e., $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$ and $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$ – and then constructing piecewise-linear approximations of these functions, with help of SOS2. By using this alternative

formulation and eliminating the need to employ McCormick's envelopes, the user may arbitrate the accuracy of the approximation of the generation currents (which is not possible when McCormick's envelopes are used). This alternative formulation is presented in Appendix C (section 9.2). Yet, it should be kept in mind that enhancing the accuracy of the piecewise-linear approximation by augmenting the number of evaluation points may result in additional computational requirements. This matter will be discussed further in 5.2.3 of this dissertation, in which both the formulation presented above and the formulation that does not employ McCormick's envelopes are used in the solution of a case study.

4.2.1.3 Loads

In the following subsections, *constant-power*, *constant-current* and *constant-impedance* loads are treated separately – these types of loads are those that pertain respectively to the sets Ω_{PCTE} , Ω_{ICTE} and Ω_{ZCTE} . For each type of load, separate subsections will deal with loads *that cannot be shed* and loads *that can be shed*.

At this point, it is necessary to remember the definition of the following sets:

- Ω_{SHED} Set of all buses to which loads that can be shed are connected;
- Ω_{NSHD} Set of all buses to which loads that cannot be shed are connected.

4.2.1.3.1 Constant-power loads that cannot be shed

Equations (9) and (10) of section 2.2.1.3.1 may be used to define the currents demanded by constant-power loads that cannot be shed. These constraints are reproduced below, for the sake of clarity:

$$I_{d,k}^{re} = \xi_k \cdot d_k^P + \zeta_k \cdot d_k^Q, \forall k \in \{\Omega_{PCTE} \cap \Omega_{NSHD}\} \quad (108)$$

$$I_{d,k}^{im} = \zeta_k \cdot d_k^P - \xi_k \cdot d_k^Q, \forall k \in \{\Omega_{PCTE} \cap \Omega_{NSHD}\} \quad (109)$$

The constraints needed for defining the auxiliary decision variables ξ_k and ζ_k have already been defined.

4.2.1.3.2 Constant-power loads that can be shed

As indicated in subsection 2.2.2.1, load shedding is considered to be a discrete decision: the load at bus k will be considered to be either energized ($\rho_k = 0$) or de-energized ($\rho_k = 1$). When the load at bus k is shed, it is obviously necessary to ensure that the associated currents will be forcefully set to zero.

Thus, the following disjunctive constraints may be used to model loads of the constant-power type that may be shed:

$$M_k^{D,re,PCTE,1} \cdot \rho_k \leq I_{d,k}^{re} - \xi_k \cdot d_k^P - \zeta_k \cdot d_k^Q \leq M_k^{D,re,PCTE,2} \cdot \rho_k$$

$$\forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (110)$$

$$M_k^{D,re,PCTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{re} \leq M_k^{D,re,PCTE,4} \cdot (1 - \rho_k)$$

$$\forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (111)$$

$$M_k^{D,im,PCTE,1} \cdot \rho_k \leq I_{d,k}^{im} - \zeta_k \cdot d_k^P + \xi_k \cdot d_k^Q \leq M_k^{D,im,PCTE,2} \cdot \rho_k$$

$$\forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (112)$$

$$M_k^{D,im,PCTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{im} \leq M_k^{D,im,PCTE,4} \cdot (1 - \rho_k)$$

$$\forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (113)$$

where:

$$M_k^{D,re,PCTE,1} ; M_k^{D,re,PCTE,2} ; M_k^{D,re,PCTE,3} ; M_k^{D,re,PCTE,4}$$

$$M_k^{D,im,PCTE,1} ; M_k^{D,im,PCTE,2} ; M_k^{D,im,PCTE,3} ; M_k^{D,im,PCTE,4}$$

Disjunctive constants for the disjunctive constraints employed for modeling shedding of loads of the constant-power type.

Section 4.3.1 will deal with the definition of these disjunctive constants.

The constraints needed for defining the auxiliary decision variables ξ_k and ζ_k have been already defined.

4.2.1.3.3 Constant-current loads that cannot be shed

Equations (11) and (12) of section 2.2.1.3.2 may be used to define the currents demanded by constant-current loads that cannot be shed. These equations are reproduced below, with slight modifications regarding the set of buses for which the constraints are defined:

$$I_{d,k}^{re} = \eta_k \cdot d_k^P + \kappa_k \cdot d_k^Q, \quad \forall k \in \{\Omega_{ICTE} \cap \Omega_{NSHD}\} \quad (114)$$

$$I_{d,k}^{im} = \kappa_k \cdot d_k^P - \eta_k \cdot d_k^Q, \quad \forall k \in \{\Omega_{ICTE} \cap \Omega_{NSHD}\} \quad (115)$$

It is necessary to present the constraints needed for the definition of the auxiliary decision variables η_k and κ_k . As seen in section 2.2.1.3.2, the auxiliary decision variables η_k and κ_k represent non-convex, non-linear functions of V_k^{re} and V_k^{im} . For the MILP reformulation of the ACOPF, piecewise-linear approximations of these non-convex, non-linear functions will be employed. Using the technique based on the construction of SOS2, the following vector equation may be employed for the definition of η_k and κ_k :

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \lambda_k^{r,s} \cdot \begin{bmatrix} \hat{\eta}_k^{r,s} \\ \hat{\kappa}_k^{r,s} \end{bmatrix} = \begin{bmatrix} \eta_k \\ \kappa_k \end{bmatrix}, \quad \forall k \in \Omega_{ICTE} \quad (116)$$

where:

$\hat{\eta}_k^{r,s}$ Evaluated values of function $\eta_k(V_k^{re}, V_k^{im})$, for bus k ;

$\hat{\kappa}_k^{r,s}$ Evaluated values of function $\kappa_k(V_k^{re}, V_k^{im})$, for bus k .

The reader will notice that, at this point, it is only necessary to define the vector equation corresponding to the *function row* of the piecewise-linearization, as all other necessary constraints have already been defined in section 4.2.1.2.1. A simple verification of the equations presented in section 4.2.1.2.1 will indicate that the sets for which equations (78) to (85) have been defined already include the set Ω_{ICTE} .

It is also clear that the constraint corresponding to equation (116) is defined not only for the loads in $\{\Omega_{ICTE} \cap \Omega_{NSHD}\}$, but to all loads of the current-type.

Section 4.3.2 deals with the definition of the *evaluated values* $\hat{\eta}_k^{r,s}$ and $\hat{\kappa}_k^{r,s}$.

4.2.1.3.4 Constant-current loads that can be shed

As load shedding is considered to be a discrete decision, the following disjunctive constraints may be used for modeling loads of the constant-current type that may be shed:

$$M_k^{D,ICTE,1} \cdot \rho_k \leq I_{d,k}^{re} - \eta_k \cdot d_k^P - \kappa_k \cdot d_k^Q \leq M_k^{D,ICTE,2} \cdot \rho_k, \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (117)$$

$$M_k^{D,ICTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{re} \leq M_k^{D,ICTE,4} \cdot (1 - \rho_k), \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (118)$$

$$M_k^{D,ICTE,1} \cdot \rho_k \leq I_{d,k}^{im} - \kappa_k \cdot d_k^P + \eta_k \cdot d_k^Q \leq M_k^{D,ICTE,2} \cdot \rho_k, \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (119)$$

$$M_k^{D,ICTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{im} \leq M_k^{D,ICTE,4} \cdot (1 - \rho_k), \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (120)$$

where:

$$M_k^{D,re,ICTE,1} ; M_k^{D,re,ICTE,2} ; M_k^{D,re,ICTE,3} ; M_k^{D,re,ICTE,4}$$

$$M_k^{D,im,ICTE,1} ; M_k^{D,im,ICTE,2} ; M_k^{D,im,ICTE,3} ; M_k^{D,im,ICTE,4}$$

Disjunctive constants for the disjunctive constraints employed for modeling shedding of loads of the constant-current type.

Section 4.3.1 will deal with the definition of the disjunctive constraints.

The constraints needed for defining the auxiliary decision variables η_k and κ_k have already been defined.

4.2.1.3.5 Constant-impedance loads that cannot be shed

Equations (15) and (16) of section 2.2.1.3.3 may be used to define the currents demanded by constant-impedance loads that cannot be shed. These equations are reproduced below, with slight modifications regarding the set of buses for which the constraints are defined:

$$I_{d,k}^{re} = V_k^{re} \cdot \frac{R_k^l}{|Z_k^l|^2} + V_k^{im} \cdot \frac{X_k^l}{|Z_k^l|^2}, \forall k \in \{\Omega_{ZCTE} \cap \Omega_{NSHD}\} \quad (121)$$

$$I_{d,k}^{im} = V_k^{im} \cdot \frac{R_k^l}{|Z_k^l|^2} - V_k^{re} \cdot \frac{X_k^l}{|Z_k^l|^2}, \forall k \in \{\Omega_{ZCTE} \cap \Omega_{NSHD}\} \quad (122)$$

Equations (121) and (122) are linear and can be readily incorporated to a MILP.

4.2.1.3.6 Constant-impedance loads that can be shed

As load shedding is considered to be a discrete decision, the following disjunctive constraints may be used for modeling loads of the constant-impedance type that can be shed:

$$M_k^{D,ZCTE,1} \cdot \rho_k \leq I_{d,k}^{re} - V_k^{re} \cdot \frac{R_k^l}{|Z_k^l|^2} - V_k^{im} \cdot \frac{X_k^l}{|Z_k^l|^2} \leq M_k^{D,ZCTE,2} \cdot \rho_k$$

$$, \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (123)$$

$$M_k^{D,ZCTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{re} \leq M_k^{D,ZCTE,4} \cdot (1 - \rho_k)$$

$$, \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (124)$$

$$M_k^{D,ZCTE,1} \cdot \rho_k \leq I_{d,k}^{im} - V_k^{im} \cdot \frac{R_k^l}{|Z_k^l|^2} + V_k^{re} \cdot \frac{X_k^l}{|Z_k^l|^2} \leq M_k^{D,ZCTE,2} \cdot \rho_k$$

$$, \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (125)$$

$$M_k^{D,ZCTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{im} \leq M_k^{D,ZCTE,4} \cdot (1 - \rho_k)$$

$$, \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (126)$$

where:

$$M_k^{D,re,ZCTE,1} ; M_k^{D,re,ZCTE,2} ; M_k^{D,re,ZCTE,3} ; M_k^{D,re,ZCTE,4}$$

$$M_k^{D,im,ZCTE,1} ; M_k^{D,im,ZCTE,2} ; M_k^{D,im,ZCTE,3} ; M_k^{D,im,ZCTE,4}$$

Disjunctive constants for the disjunctive constraints employed for modeling shedding of loads of the constant-impedance type.

Section 4.3.1 will deal with the definition of the disjunctive constraints mentioned above.

4.2.1.4 Operating limits

4.2.1.4.1 Bounds on bus voltage magnitudes

The magnitude of the voltage at bus k is a non-linear, non-convex function of the real and imaginary components of the voltage at this bus, as indicated in section 2.2.1.4.1. It is thus necessary to obtain an approximation of this decision variable –

which will be done with help of the technique presented in section 3.2 of this dissertation. The following equation may be employed for the definition of V_k :

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \lambda_k^{r,s} \cdot \hat{V}_k^{r,s} = V_k \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (127)$$

where:

$\hat{V}_k^{r,s}$ Evaluated values of function $V_k(V_k^{re}, V_k^{im})$, for bus k .

The equation above corresponds to the *function row* for the piecewise-linear approximation. The reader will recall that all other necessary constraints have already been defined in section 4.2.1.2.1.

After using (127) to obtain an approximation of V_k , the following constraint may be used to impose bounds on this variable:

$$\underline{V}_k \leq V_k \leq \bar{V}_k \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (128)$$

4.2.1.4.2 Bounds on the magnitude of branch currents

Analogously to what has been seen in the previous section, the magnitude of the current flowing through branch km is a non-linear, non-convex function of its real and imaginary components. Thus, an approximation of this non-convex, non-linear function will be required for the MILP reformulation of the ACOPF.

When constructing a piecewise-linear approximation of the bus voltage magnitude, the fact that there were several other decision variables that were non-convex, non-linear functions of the real and imaginary components of the bus voltage at each bus was taken advantage of. Taking that into account, it was only necessary to define constraints referring to the *function row* of piecewise-linear approximation technique described in section 3.2.

This is not the case for the real and imaginary components of the branch currents – there are no other non-linear functions of these variables.

As the only function of the real and imaginary components of the branch currents that will need to be approximated is the magnitude of the corresponding

complex quantity, the fact that the function $I_{km} = \sqrt{(I_{km}^{re})^2 + (I_{km}^{im})^2}$ is symmetric about the origin can be used for reducing the number of *evaluation points* needed to obtain the piecewise-linear approximation.

In order to do that, it is first necessary to define auxiliary variables that will be at least as high as the modulus of the components I_{km}^{re} and I_{km}^{im} . This may be done with help of the following constraints:

$$l_{km}^{re} \geq I_{km}^{re} \quad , \forall k \in \Psi_C \quad (129)$$

$$l_{km}^{re} \geq -I_{km}^{re} \quad , \forall k \in \Psi_C \quad (130)$$

$$l_{km}^{im} \geq I_{km}^{im} \quad , \forall k \in \Psi_C \quad (131)$$

$$l_{km}^{im} \geq -I_{km}^{im} \quad , \forall k \in \Psi_C \quad (132)$$

where:

l_{km}^{re} Auxiliary variable that is at least as high as the modulus of I_{km}^{re} , for branch km ;

l_{km}^{im} Auxiliary variable that is at least as high as the modulus of I_{km}^{im} , for branch km .

The reader will notice that, given that l_{km}^{re} and l_{km}^{im} are at least as high as the modulus of I_{km}^{re} and I_{km}^{im} , the square root of the sum of the squared values of these auxiliary variables will always be at least as high as the square root of the sum of the squared values of the current components. Thus, enforcing bounds on the former square root will result in the latter being bounded.

It is thus necessary to obtain an approximation of $l_{km} = \sqrt{(l_{km}^{re})^2 + (l_{km}^{im})^2}$. This can be done by building a piecewise-linear approximation of this function, with help of the technique presented in section 3.2. This piecewise-linear approximation can be obtained with help of the following equations:

$$\sum_{r \in \Pi^{re}} \sum_{s \in \Pi^{im}} \omega_{km}^{r,s} \cdot \hat{l}_{km}^{r,s} = l_{km} \quad , \forall km \in \Psi_C \quad (133)$$

$$\sum_{r \in \Pi^{re}} \sum_{s \in \Pi^{im}} \omega_{km}^{r,s} \cdot \begin{bmatrix} \hat{l}_{km}^{re,r} \\ \hat{l}_{km}^{im,s} \end{bmatrix} = \begin{bmatrix} l_{km}^{re} \\ l_{km}^{im} \end{bmatrix} \quad , \forall km \in \Psi_C \quad (134)$$

$$\sum_{r \in \Pi^{re}} \sum_{s \in \Pi^{im}} \omega_{km}^{r,s} = 1 \quad , \forall km \in \Psi_C \quad (135)$$

where:

- Π^{re} Set of indices for *evaluation points* $\hat{l}_{km}^{re,r}$ and associated variables;
- Π^{im} Set of indices for *evaluation points* $\hat{l}_{km}^{im,s}$ and associated variables;
- $\hat{l}_{km}^{re,r}$ Evaluation points of l_{km}^{re} , for branch km ;
- $\hat{l}_{km}^{im,s}$ Evaluation points of l_{km}^{im} , for branch km ;
- $\hat{l}_{km}^{r,s}$ Evaluated values of function $l_{km} = \sqrt{(l_{km}^{re})^2 + (l_{km}^{im})^2}$, for branch km ;
- $\omega_{km}^{r,s}$ Weights for constructing piecewise-linear approximation of l_{km} , for branch km ;
- l_{km} Auxiliary variable that is at least as high as I_{km} , for branch km .

Section 4.3.2 deals with the definition of the *evaluation points* and *evaluated values* $\hat{l}_{km}^{re,r}$, $\hat{l}_{km}^{im,s}$ and $\hat{l}_{km}^{r,s}$.

The following constraints ensure that the variables $\omega_{km}^{r,s}$ form a SOS2:

$$\sum_{r \in \Pi^{re}} t_{km}^r = 1 \quad , \forall km \in \Psi_C \quad (136)$$

$$\lambda_{km}^{1,s} \leq t_{km}^1 \quad , \forall s \in \Pi^{im}, km \in \Psi_C \quad (137)$$

$$\lambda_{km}^{r,s} \leq t_{km}^{r-1} + t_{km}^r \quad , \forall r \in \{\Pi^{re} \setminus \{1\}\}, s \in \Pi^{im}, km \in \Psi_C \quad (138)$$

$$\sum_{s \in \Pi^{im}} u_{km}^s = 1 \quad , \forall k \in \Psi_C \quad (139)$$

$$\lambda_{km}^{r,1} \leq u_{km}^1 \quad , \forall r \in \Pi^{re}, km \in \Psi_C \quad (140)$$

$$\lambda_{km}^{r,s} \leq u_{km}^{s-1} + u_{km}^s \quad , \forall r \in \Pi^{re}, s \in \{\Pi^{im} \setminus \{1\}\}, km \in \Psi_C \quad (141)$$

where:

$t_{km}^r ; u_{km}^s$ Auxiliary binary decision variables.

After obtaining an approximation of l_{km} , the following constraint may be used for bounding this variable (and indirectly bounding the magnitude of the current flowing through branch km):

$$l_{km} \leq \bar{l}_{km} \quad , \forall km \in \Psi_C \quad (142)$$

4.2.1.4.3 Bounds on active and reactive power output of generators

The constraints of section 2.2.1.4.3, reproduced below for the sake of clarity, may be employed for bounding the active and reactive power output of generators:

$$\underline{g}_k^P \leq g_k^P \leq \overline{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (143)$$

$$\underline{g}_k^Q \leq g_k^Q \leq \overline{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (144)$$

4.2.1.5 Voltage reference buses

The constraints of section 2.2.1.5, reproduced below for the sake of clarity, may be employed for specifying the real and imaginary components of the voltage of buses pertaining to Ω_{REF} . For applications in which the voltage magnitude of these reference buses is fixed, the following constraints apply:

$$V_k^{re} = V_k^{ref} \cdot \cos\theta_k^{ref} \quad , \forall k \in \Omega_{REF} \quad (145)$$

$$V_k^{im} = V_k^{ref} \cdot \sin\theta_k^{ref} \quad , \forall k \in \Omega_{REF} \quad (146)$$

For applications in which the magnitude of the voltage at reference buses are decision variables of the ACOPF, the following constraints apply:

$$V_k^{re} = V_k \cdot \cos\theta_k^{ref} \quad , \forall k \in \Omega_{REF} \quad (147)$$

$$V_k^{im} = V_k \cdot \sin\theta_k^{ref} \quad , \forall k \in \Omega_{REF} \quad (148)$$

Again, it is important to emphasize that, for applications in which the voltage magnitudes of the buses in Ω_{REF} are considered decision variables, it is necessary to enforce the corresponding bounds by using the following constraint:

$$\underline{V}_k \leq V_k \leq \overline{V}_k \quad , \forall k \in \Omega_{REF} \quad (149)$$

4.2.1.6 Slack buses and buses without generators and/or loads

The constraints of section 2.2.1.6, reproduced below for the sake of clarity, may be used to ensure that the load/generation currents of buses to which no loads/generators connect are set to zero. The reader will notice that the generator currents of all buses in the set Ω_{SLACK} may assume any given value.

$$I_{d,k}^{re} = I_{d,k}^{im} = 0, \forall k \in \{\Omega_B \setminus \Omega_{LOAD}\} \quad (150)$$

$$I_{g,k}^{re} = I_{g,k}^{im} = 0, \forall k \in \{\Omega_B \setminus \{\Omega_{GEN} \cup \Omega_{SLACK}\}\} \quad (151)$$

4.2.1.7 Radiality constraints

If it is required to ensure that the distribution network is radial, the constraints presented in this section are to be added to the MILP formulation of the ACOPF for distribution systems. Reference [38] introduced a formulation of radiality constraints based on ensuring that the distribution network consists of a spanning tree that originates from the *root node*. This approach, which is based on using binary decision variables to impose a particular structure to the distribution system, can be readily incorporated to a mixed-integer program and will be used in this dissertation.

In the following subsections, three slightly different approaches for the formulation of the radiality constraints are presented. The three approaches ensure that all nodes that are connected to the network are arranged within a radial structure – however, the approaches differ in the specification of which nodes *must* be connected to the network and which *can be removed* from it. In order to make it clear that the three approaches differ only with respect to this aspect, they will be referred to as *connectivity approaches* to the radiality constraints.

At this point, the reader may want to consider why it is necessary to define more than one *connectivity approach*. Depending on specific characteristics of the distribution system operations or expansion planning application under consideration, it may be necessary to remove from the network some (or all) of the buses to which loads that have been shed and/or generators that have been curtailed connect. In previous

sections of this dissertation, reference has been made to the fact that, as of this writing, the most common approach to disconnect elements of the distribution system in case of emergencies is to maneuver switches and de-energize entire segments of the distribution network. Thus, depending on the application at hand, it may be in the interest of the distribution system planner to ensure that one load can be disconnected only if all circuits connected to it are de-energized. For some other applications, this may not be necessary.

Keeping this in mind, one of the three following *connectivity approaches* may be chosen while formulating the radiality constraints:

- (i) For this first approach, all buses of the distribution system, which have been defined in the input data for the ACOPF, *must* be connected to the network at all times – even if the loads and/or generators connected to it are curtailed.
- (ii) For the second approach, it is considered that the load and/or generator at a bus can only be de-energized (shed and curtailed, respectively) if all circuits that connect to that bus are removed from the network (i.e., all circuits must have their status changed to *inactive*).
- (iii) For the third approach, it is considered that the buses to which loads that are shed and generators that are curtailed, as well as all buses that do have any potential injections (i.e., those that are not reference or slack buses and to which no loads or generators connect), may or may not be disconnected from the network, according to the distribution system planner decision. Thus, the optimality of the decision is the only criterion that dictates if these buses will be connected to or disconnected from the network.

The mathematical formulation corresponding to the three basic connectivity approaches listed above is presented in the following subsections. In subsection 5.2.1 of this dissertation, an example of the application of each of these three approaches is presented.

It is worth pointing out that, despite the fact that the three approaches are presented in different subsections for the sake of didactics, it is possible to combine them within a single optimization problem, utilizing different connectivity approaches for different buses of the distribution system.

4.2.1.7.1 Connectivity approach (i)

For approach (i), all buses of the distribution system, which have been defined in the input data for the ACOPF, must be connected to the network at all times.

The formulation of the radiality constraints corresponding to this connectivity approach corresponds exactly to that proposed in [38]. In order to ensure radiality, it suffices to determine that every bus in the network has exactly one *parent bus*, except for the *root bus*. Each spanning tree in the distribution system (each islanded, radial system) originates from a *root bus*, and none of the *root buses* have parents. The following set of constraints may be used to impose this particular structure to the distribution system:

$$v_{km}^k + v_{km}^m = 1 \quad , \forall km \in \{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\} \quad (152)$$

$$v_{km}^k + v_{km}^m = \sigma_{km} \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (153)$$

$$[\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] = 1 \quad , \forall k \in \{\Omega_B \setminus \Omega_{ROOT}\} \quad (154)$$

$$v_{km}^m = 0 \quad , \forall km \in \{\Psi_C | k \in \Omega_{ROOT}\} \quad (155)$$

$$v_{km}^k = 0 \quad , \forall km \in \{\Psi_C | m \in \Omega_{ROOT}\} \quad (156)$$

where:

v_{km}^k Binary variable associated to line km that assumes the value $v_{km}^k = 1$ if bus k is the *parent* of bus m , and that assumes the value $v_{km}^k = 0$ if bus m is the *parent* of bus k ;

Ω_{ROOT} Set of buses chosen as *root buses*. The number of *root buses* must equal the number of allowed islands in the system.

The reader will notice that equations (152) and (153) ensure that, for every active branch km , either k is the parent of bus m or m is the parent of bus k . Equation (154) ensures that every bus in the system, except the *root buses*, has one and only one parent. Equations (155) and (156) ensure that, for every branch km that includes one root bus at one its extremities, the bus that is not the root cannot be a *parent bus* – i.e., the root bus will always be a parent bus if there is an active circuit connected to it [38].

As pointed out in [38], constraint (153) suffices for ensuring that the variable σ_{km} can only assume the values 0 or 1, even if σ_{km} is defined as a continuous (rather than a binary) variable. Therefore, whenever this constraint is added to the MILP formulation of the ACOPF, σ_{km} may be defined as a continuous (rather than a binary) variable. Nonetheless, the reader will recall that the MILP reformulation of the ACOPF for distribution systems presented in this dissertation may be applied to distribution systems that are operated in a radial fashion *or* in a meshed fashion – thus, whenever a radial operation is not required, constraint (153) will not be part of the MILP problem. This is the reason why, in previous sections, σ_{km} has always been defined as *binary* decision variable. However, as pointed out in this paragraph, whenever the radiality constraints are added to the model, σ_{km} may be defined as a continuous variable.

It is worth pointing out that it is not required that Ω_{ROOT} coincide with the set Ω_{REF} or to the set Ω_{SLACK} . It should be kept in mind that Ω_{SLACK} may be an empty set depending on the application, but the cardinality of the set Ω_{REF} must always equal the number of potentially islanded systems in the network, with one and only one reference voltage bus defined for every island. Thus, in many applications, it may be in the interest of the distribution system engineer to define $\Omega_{ROOT} = \Omega_{REF}$, despite of this not being compulsory.

4.2.1.7.2 Connectivity approach (ii)

For connectivity approach (ii), it is considered that the load and/or the generator at a bus can only be de-energized if all circuits that connect to that bus are removed from the network.

In order to model this condition, it is necessary to modify some of the constraints proposed in [38]. The modifications proposed in this dissertation will be presented in the following.

The first two constraints of the previous section, which ensure that every active branch has one and only one parent bus in its extremities, are not modified:

$$v_{km}^k + v_{km}^m = 1, \forall km \in \{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\} \quad (157)$$

$$v_{km}^k + v_{km}^m = \sigma_{km}, \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (158)$$

However, the constraints that control which buses in the network must have *parents* need to be modified. In this dissertation, these will be referred to as *parenthood* constraints. The formulation of these constraints for buses with loads that can be shed, but to which no curtailable generators connect, is indicated below:

$$\begin{aligned} [\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] &= (1 - \rho_k) \\ &, \forall k \in \{\Omega_{SHED} \setminus \Omega_{CURT}\} \end{aligned} \quad (159)$$

$$v_{km}^k \leq (1 - \rho_k) \quad , \forall km \in \{\Psi_C | k \in \{\Omega_{SHED} \setminus \Omega_{CURT}\}\} \quad (160)$$

$$v_{km}^m \leq (1 - \rho_k) \quad , \forall km \in \{\Psi_C | m \in \{\Omega_{SHED} \setminus \Omega_{CURT}\}\} \quad (161)$$

Equation (159) ensures that, if a given load is shed ($\rho_k = 1$), it does not have any *parent*; and if the bus is not shed ($\rho_k = 0$), it has exactly one parent. Equations (160) and (161) ensure that buses with loads that have been shed cannot be the parents of any other buses; but buses with loads that have not been shed may be the parents of other buses. It is thus clear that a bus with a load that has been shed will not have any parents and it will not be the parent to any other buses, meaning that this bus will be disconnected from the network.

A set of analogous constraints are defined for buses that have curtailable generators, but no loads that can be shed (i.e., buses in $\{\Omega_{CURT} \setminus \Omega_{SHED}\}$). The only difference is that the binary variable that controls generation curtailment is τ_k , and not ρ_k :

$$\begin{aligned} [\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] &= (1 - \tau_k) \\ &, \forall k \in \{\Omega_{CURT} \setminus \Omega_{SHED}\} \end{aligned} \quad (162)$$

$$v_{km}^k \leq (1 - \tau_k) \quad , \forall km \in \{\Psi_C | k \in \{\Omega_{CURT} \setminus \Omega_{SHED}\}\} \quad (163)$$

$$v_{km}^m \leq (1 - \tau_k) \quad , \forall km \in \{\Psi_C | m \in \{\Omega_{CURT} \setminus \Omega_{SHED}\}\} \quad (164)$$

It is now necessary to model buses that have loads that can be shed *and* generators that can be curtailed (i.e., buses in $\{\Omega_{SHED} \cap \Omega_{CURT}\}$). As any of these elements may only be de-energized if the bus is entirely removed from the network, shedding the load necessarily requires curtailing the generator, and vice-versa. Thus, the following set of constraints may be used:

$$\tau_k = \rho_k \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (165)$$

$$\begin{aligned} [\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] &= (1 - \rho_k) \\ & , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \end{aligned} \quad (166)$$

$$v_{km}^k \leq (1 - \rho_k) \quad , \forall km \in \{\Psi_C | k \in \{\Omega_{SHED} \cap \Omega_{CURT}\}\} \quad (167)$$

$$v_{km}^m \leq (1 - \rho_k) \quad , \forall km \in \{\Psi_C | m \in \{\Omega_{SHED} \cap \Omega_{CURT}\}\} \quad (168)$$

The reader will notice that, if the connectivity approach (ii) is employed, for all buses in $\{\Omega_{SHED} \cap \Omega_{CURT}\}$, is possible to substitute τ_k by ρ_k in every constraint of section 4.2.1.2.2, dropping the binary variable τ_k altogether from the formulation of the MILP problem. This is not done here, however, for the sake of simplicity.

Now that all buses with loads that can be shed and curtailable generation have been treated, the *parenthood* constraints for the remainder of the buses in the network are presented:

$$\begin{aligned} [\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] &= 1 \\ & , \forall k \in \{\Omega_B \setminus \{\Omega_{SHED} \cup \Omega_{CURT} \cup \Omega_{ROOT}\}\} \end{aligned} \quad (169)$$

For the formulation above, we consider that the sets Ω_{SHED} and Ω_{CURT} are defined in such a way that there are no buses that have either loads that cannot be shed and curtailable generators, of non-curtailable generators and loads that can be shed.

As in connectivity approach (i), the root buses do not have any parents. Also, the root buses are necessarily the parents of all nodes directly connected to them through active circuits, as indicated by the following constraints, whose formulation remains unchanged:

$$v_{km}^m = 0 \quad , \forall km \in \{\Psi_C | k \in \Omega_{ROOT}\} \quad (170)$$

$$v_{km}^k = 0 \quad , \forall km \in \{\Psi_C | m \in \Omega_{ROOT}\} \quad (171)$$

4.2.1.7.3 Connectivity approach (iii)

For the third connectivity approach, it is considered that the buses to which loads that are shed and generators that are curtailed, as well as all buses that do have any fixed or curtailable injections, may or may not be disconnected from the network, according exclusively to the distribution system planner decision. That is to say, the only criterion that determines that one of these buses will or will not be connected to the network is the impact of this decision on the objective function of the optimization problem.

As in approach (ii), the following constraints are exactly equal to those presented in section 4.2.1.7.1:

$$v_{km}^k + v_{km}^m = 1 \quad , \forall km \in \{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\} \quad (172)$$

$$v_{km}^k + v_{km}^m = \sigma_{km} \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (173)$$

Yet, the *parenthood* constraints must also be modified in approach (iii). The modifications proposed in this dissertation will be presented in the following. The following constraints apply to the set of buses to which loads that can be shed connect, but to which no curtailable generators connect (i.e., buses in $\{\Omega_{SHED} \setminus \Omega_{CURT}\}$):

$$\beta_k \geq (1 - \rho_k) \quad , \forall k \in \{\Omega_{SHED} \setminus \Omega_{CURT}\} \quad (174)$$

$$[\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] = \beta_k \quad , \forall k \in \{\Omega_{SHED} \setminus \Omega_{CURT}\} \quad (175)$$

$$v_{km}^k \leq \beta_k \quad , \forall km \in \{\Psi_C | k \in \{\Omega_{SHED} \setminus \Omega_{CURT}\}\} \quad (176)$$

$$v_{km}^m \leq \beta_k \quad , \forall km \in \{\Psi_C | m \in \{\Omega_{SHED} \setminus \Omega_{CURT}\}\} \quad (177)$$

where:

β_k Binary decision variable that models the decision to disconnect a bus k from the system: if $\beta_k = 0$, the bus is disconnected from the system, if $\beta_k = 1$, the bus is connected to the system.

A few words on the logical implications of the constraints defined above are in order at this point:

- It is clear that, if the load at bus k has *not* been shed, then $\beta_k \geq 1$. Since β_k is binary and can only assume the values 0 or 1, $\beta_k \geq 1$ leads to $\beta_k = 1$, meaning that bus k is connected to the system (it will have one and only one parent, and it may or may not be the parent to other buses).
- If the load at bus k has been shed, β_k may be entirely free to assume the values $\beta_k = 0$ or $\beta_k = 1$. If $\beta_k = 1$, the situation described above is valid. If $\beta_k = 0$, bus k will have no parents and will not be the parent to any other buses in the network – thus, bus k has been removed from the network.

From the explanation above, it is clear that, given that the load at bus k has been shed, the decision to remove or not a bus from the network is dictated only by its impact on the objective function.

A set of analogous constraints are defined for buses in $\{\Omega_{CURT} \setminus \Omega_{SHED}\}$:

$$\beta_k \geq (1 - \tau_k) \quad , \forall k \in \{\Omega_{CURT} \setminus \Omega_{SHED}\} \quad (178)$$

$$[\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] = \beta_k \quad , \forall k \in \{\Omega_{CURT} \setminus \Omega_{SHED}\} \quad (179)$$

$$v_{km}^k \leq \beta_k \quad , \forall km \in \{\Psi_C | k \in \{\Omega_{CURT} \setminus \Omega_{SHED}\}\} \quad (180)$$

$$v_{km}^m \leq \beta_k \quad , \forall km \in \{\Psi_C | m \in \{\Omega_{CURT} \setminus \Omega_{SHED}\}\} \quad (181)$$

For buses that have *both* loads that can be shed *and* curtailable generators, the situation is slightly more complex. A bus in set $\{\Omega_{SHED} \cap \Omega_{CURT}\}$ may be only disconnected from the network if the load has been shed ($\rho_k = 1$) and the generator has been curtailed ($\tau_k = 1$). In order to check if this condition is met, an auxiliary, continuous decision variable α_k is introduced to the problem, and the following constraints apply:

$$\alpha_k \leq \tau_k \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (182)$$

$$\alpha_k \leq \rho_k \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (183)$$

$$\alpha_k \geq (\tau_k + \rho_k) - 1 \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (184)$$

$$\alpha_k \leq 1 \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (185)$$

where:

α_k Continuous (non-negative) decision that assumes the value $\alpha_k = 1$ if and only if $\rho_k = 1$ and $\tau_k = 1$; and assumes the value $\alpha_k = 0$ for all other combinations of the binary variables ρ_k and τ_k .

Having defined the auxiliary variable above, the *parenthood* constraints for the buses in $\{\Omega_{SHED} \cap \Omega_{CURT}\}$ may be formulated as:

$$\beta_k \geq (1 - \alpha_k) \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (186)$$

$$[\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] = \beta_k \quad , \forall k \in \{\Omega_{SHED} \cap \Omega_{CURT}\} \quad (187)$$

$$v_{km}^k \leq \beta_k \quad , \forall km \in \{\Psi_C | k \in \{\Omega_{SHED} \cap \Omega_{CURT}\}\} \quad (188)$$

$$v_{km}^m \leq \beta_k \quad , \forall km \in \{\Psi_C | m \in \{\Omega_{SHED} \cap \Omega_{CURT}\}\} \quad (189)$$

Now, it is necessary to model the fact that every bus to which no injections connect (i.e., buses that have no loads or generators, and that are not slack buses) and that are not reference buses can be removed from the network if desired. This is done with help of the following set of constraints:

$$[\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] = \beta_k \quad , \forall k \in \{\Omega_B \setminus \{\Omega_{LOAD} \cup \Omega_{GEN} \cup \Omega_{SLACK} \cup \Omega_{REF} \cup \Omega_{ROOT}\}\} \quad (190)$$

$$v_{km}^k \leq \beta_k \quad , \forall km \in \{\Psi_C | k \in \{\Omega_B \setminus \{\Omega_{LOAD} \cup \Omega_{GEN} \cup \Omega_{SLACK} \cup \Omega_{REF} \cup \Omega_{ROOT}\}\}\} \quad (191)$$

$$v_{km}^m \leq \beta_k \quad , \forall km \in \{\Psi_C | m \in \{\Omega_B \setminus \{\Omega_{LOAD} \cup \Omega_{GEN} \cup \Omega_{SLACK} \cup \Omega_{REF} \cup \Omega_{ROOT}\}\}\} \quad (192)$$

Finally, it is necessary to ensure that the buses that do not pertain to any of the sets defined above are always connected to the network. This is done by defining the following constraints:

$$[\sum_{ij \in \Psi_C, i=k} v_{ij}^j] + [\sum_{ij \in \Psi_C, j=k} v_{ij}^i] = 1 \quad , \forall k \in \{\{\Omega_{LOAD} \cup \Omega_{GEN} \cup \Omega_{SLACK} \cup \Omega_{REF}\} \setminus \{\{\Omega_{SHED} \cap \Omega_{CURT}\} \cup \Omega_{ROOT}\}\} \quad (193)$$

Finally, the following constraints ensure that the root buses are necessarily the parents of all nodes directly connected to them:

$$v_{km}^m = 0, \forall km \in \{\Psi_C | k \in \Omega_{ROOT}\} \quad (194)$$

$$v_{km}^k = 0, \forall km \in \{\Psi_C | m \in \Omega_{ROOT}\} \quad (195)$$

4.2.2 Objective functions for selected distribution system operations and expansion planning applications

In this section, the objective functions presented in section 2.2.2 are revisited. At this point, the binary variables that represent discrete decisions have already been presented to the reader, allowing a better comprehension of the mathematical formulation of the objective functions presented below, as well as of the MILP reformulation of the ACOPF as a whole.

The majority of the objective functions presented at section 2.2.2 can be readily factored into mixed-integer linear problems. Due to that, most of the equations of section 2.2.2 will be simply reproduced below with no further manipulation.

4.2.2.1 *Minimization of costs of load shedding*

As discussed in sections 2.2.2.1 and 4.2.1.3, the focus of this dissertation is on the case in which load shedding is a discrete decision. The following objective function, first introduced in section 2.2.2.1 and reproduced below for the sake of clarity, can be directly integrated to the MILP reformulation of the ACOPF:

$$z^{SHED} = \min\{\sum_{k \in \Omega_{SHED}} c_k^{SHED} \cdot d_k^P \cdot \rho_k\} \quad (196)$$

4.2.2.2 *Minimization of curtailment of non-controllable generation*

As previously discussed, the power output of generators pertaining to Ω_{CTRQ} cannot be controlled – meaning that, for these generators, g_k^P is a parameter of the

optimization problem. Consequently, the following objective function, first introduced in section 2.2.2.2 and reproduced below for the sake of clarity, can be directly integrated to the MILP reformulation of the ACOPF:

$$z^{GCRT} = \min \left\{ \sum_{k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\}} c_k^{CURT} \cdot g_k^P \cdot \tau_k \right\} \quad (197)$$

4.2.2.3 *Minimization of generation costs*

The following objective function, first introduced in section 2.2.2.3, can be directly integrated to the MILP reformulation of the ACOPF:

$$z^{GEN} = \min \left\{ \sum_{k \in \Omega_{CTRPQ}} c_k^{GEN} \cdot g_k^P \right\} \quad (198)$$

4.2.2.4 *Minimization of costs of power imports*

Section 2.2.2.4 has presented three slightly different formulations of the objective function associated with the problem of minimizing the costs of power imports from an external network. The two formulations of practical interest correspond to equations (35) and (36).

The latter of these corresponds to a linear equation that can be incorporated to a MILP without further manipulation, due to $I_{g,k}^{re}$ being the only (continuous) decision variable appearing in equation (36) – all other terms are parameters of the optimization problem. For the sake of clarity, equation (36) is reproduced below:

$$z^{IMPR} = \min \left\{ \sum_{k \in \Omega_{ITFC}} c_k^{IMPORT} \cdot V_k^{ref} \cdot I_{g,k}^{re} \right\} \quad (199)$$

However, the bilinear product of decision variables $V_k \cdot I_{g,k}^{re}$ appears in equation (35). It is obviously necessary to approximate this product before the inclusion of an objective function of this type into a mixed-integer linear program. Two alternatives for obtaining such an approximation are presented in the following subsections.

4.2.2.4.1 Approximation via McCormick's envelope

The first alternative is to substitute the bilinear product by an auxiliary variable P_k and bound this auxiliary variable within the convex envelope for the original product. This is done by adding the following constraints to the optimization problem:

$$P_k \geq \underline{V}_k \cdot I_{g,k}^{re} + V_k \cdot \underline{I}_{g,k}^{re} - \underline{V}_k \cdot \underline{I}_{g,k}^{re}, \quad \forall k \in \Omega_{ITFC} \quad (200)$$

$$P_k \geq \bar{V}_k \cdot I_{g,k}^{re} + V_k \cdot \bar{I}_{g,k}^{re} - \bar{V}_k \cdot \bar{I}_{g,k}^{re}, \quad \forall k \in \Omega_{ITFC} \quad (201)$$

$$P_k \leq \underline{V}_k \cdot I_{g,k}^{re} + V_k \cdot \bar{I}_{g,k}^{re} - \underline{V}_k \cdot \bar{I}_{g,k}^{re}, \quad \forall k \in \Omega_{ITFC} \quad (202)$$

$$P_k \leq \bar{V}_k \cdot I_{g,k}^{re} + V_k \cdot \underline{I}_{g,k}^{re} - \bar{V}_k \cdot \underline{I}_{g,k}^{re}, \quad \forall k \in \Omega_{ITFC} \quad (203)$$

where:

P_k Auxiliary (continuous) decision variable for approximating the product $V_k \cdot I_{g,k}^{re}$, for all buses k in Ω_{ITFC} ;

$\underline{V}_k ; \bar{V}_k$ Lower and upper bounds for the voltage magnitude for bus k (as mentioned in section 4.2.1.5);

$\underline{I}_{g,k}^{re} ; \bar{I}_{g,k}^{re}$ Lower and upper bounds for the real component of the *slack current* of bus k in Ω_{ITFC} .

The objective function corresponding to equation (35) may be then rewritten as:

$$z^{IMPR} = \min\{\sum_{k \in \Omega_{ITFC}} c_k^{IMPORT} \cdot P_k\} \quad (204)$$

The definition of $\underline{I}_{g,k}^{re}$ and $\bar{I}_{g,k}^{re}$, the bounds for the real component of the *slack current* of bus k in Ω_{ITFC} , is related strictly to the necessity of incorporating these bounds into the constraints of McCormick's envelope. The definition of these bounds will be dealt with in section 4.3.2.

4.2.2.4.2 Piecewise-linear approximation with the use of SOS2

If the accuracy of the approximation via McCormick's envelope is not considered satisfactory, an SOS2-based piecewise-linear approximation of $V_k \cdot I_{g,k}^{re}$ may

be used. The product is substituted by an auxiliary variable P_k , and the following constraints are added to the problem:

$$\sum_{r \in E^V} \sum_{s \in E^{I^g}} \varphi_k^{r,s} \cdot \hat{P}_k^{r,s} = P_k \quad , \forall k \in \Omega_{ITFC} \quad (205)$$

$$\sum_{r \in E^V} \sum_{s \in E^{I^g}} \varphi_k^{r,s} \cdot \begin{bmatrix} \hat{V}_k^r \\ \hat{I}_{g,k}^{re,s} \end{bmatrix} = \begin{bmatrix} V_k \\ I_{g,k}^{re} \end{bmatrix} \quad , \forall k \in \Omega_{ITFC} \quad (206)$$

$$\sum_{r \in E^V} \sum_{s \in E^{I^g}} \varphi_k^{r,s} = 1 \quad , \forall k \in \Omega_{ITFC} \quad (207)$$

$$\sum_{r \in E^V} b_k^r = 1 \quad , \forall k \in \Omega_{ITFC} \quad (208)$$

$$\varphi_k^{1,s} \leq b_k^1 \quad , \forall s \in E^{I^g}, k \in \Omega_{ITFC} \quad (209)$$

$$\varphi_k^{r,s} \leq b_k^{r-1} + b_k^r \quad , \forall r \in \{E^V \setminus \{1\}\}, s \in E^{I^g}, k \in \Omega_{ITFC} \quad (210)$$

$$\sum_{s \in E^{I^g}} c_k^s = 1 \quad , \forall k \in \Omega_{ITFC} \quad (211)$$

$$\varphi_k^{r,1} \leq c_k^1 \quad , \forall r \in E^V, k \in \Omega_{ITFC} \quad (212)$$

$$\varphi_k^{r,s} \leq c_k^{s-1} + c_k^s \quad , \forall r \in E^V, s \in \{E^{I^g} \setminus \{1\}\}, k \in \Omega_{ITFC} \quad (213)$$

where:

- E^V Set of indices for *evaluation points* \hat{V}_k^r and associated variables;
- Γ^{im} Set of indices for *evaluation points* $\hat{I}_{g,k}^s$ and associated variables;
- \hat{V}_k^r Evaluation points of voltage magnitude of bus k in Ω_{ITFC} ;
- $\hat{I}_{g,k}^{re,s}$ Evaluation points of real component of slack current of bus k in Ω_{ITFC} ;
- $\hat{P}_k^{r,s}$ Evaluated values of function $P_k(V_k, I_{g,k}^{re})$, for bus k ;
- $\varphi_k^{r,s}$ Weights for constructing piecewise-linear approximation of non-convex, non-linear function of V_k and $I_{g,k}^{re}$.
- $b_k^r ; c_k^s$ Auxiliary binary decision variables.

The objective function corresponding to equation (35) is then rewritten as:

$$z^{IMPR} = \min\{\sum_{k \in \Omega_{ITFC}} c_k^{IMPORT} \cdot P_k\} \quad (214)$$

4.2.2.5 Minimization of costs of ohmic losses

A non-linear objective function corresponding to the problem of minimization of the cost of losses has been presented in section 2.2.2.5. The non-linearities in equation (37) are associated with the terms V_k^2 , $V_k^2 \cdot (1 - \rho_k)$ and $V_k \cdot (1 - \rho_k)$, where V_k is a continuous and ρ_k a discrete decision variable.

One possible way to deal with these non-linearities is to assume that, for the range of variation of the magnitude of buses to which loads of the *constant-current type* and of the *constant-impedance type*, it suffices to approximate $V_k \approx 1$. This would result in the following approximated objective function, which can be readily incorporated to a mixed-integer linear program:

$$\begin{aligned}
z^{LOSS} = \min \{ & c_k^{LOSS} \cdot \{ \sum_{k \in \Omega_{ITFC}} V_k^{ref} \cdot I_{g,k}^{re} + \sum_{k \in \Omega_{CTRPQ}} g_k^P \\
& + \sum_{k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\}} g_k^P + \sum_{k \in \Omega_{CURT}} g_k^P \cdot (1 - \tau_k) \\
& - [\sum_{k \in \{\Omega_{PCTE} \cap \Omega_{NSHD}\}} d_k^P + \sum_{k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\}} d_k^P \cdot (1 - \rho_k)] \\
& - [\sum_{k \in \{\Omega_{ICTE} \cap \Omega_{NSHD}\}} d_k^P + \sum_{k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\}} d_k^P \cdot (1 - \rho_k)] \\
& - \left[\sum_{k \in \{\Omega_{ZCTE} \cap \Omega_{NSHD}\}} \frac{R_k^l}{|Z_k^l|^2} + \sum_{k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\}} \frac{R_k^l}{|Z_k^l|^2} \cdot (1 - \rho_k) \right] \} \} \quad (215)
\end{aligned}$$

If this approximation is not considered satisfactory, it is possible to employ a number of reformulation techniques for approximating the non-linear term V_k^2 by an auxiliary continuous decision variable μ_k (some of them allowing approximations of arbitrary accuracy), and to exactly represent the products $\mu_k \cdot (1 - \rho_k)$ and $V_k \cdot (1 - \rho_k)$. These techniques are presented in the following subsections. The MILP reformulation of the objective function for the minimization of losses is then summarized in subsection 4.2.2.5.3.

4.2.2.5.1 Approximation of V_k^2

There are a number of alternatives for obtaining an approximation of the term V_k^2 that can be employed within a mixed-integer linear program. All of the alternatives require the substitution of the term V_k^2 by an auxiliary variable, which will be referred

to as μ_k in the following. All alternatives will obviously take advantage of the fact that V_k , being a bus voltage magnitude, may only assume non-negative values in the interval $\underline{V}_k \leq V_k \leq \bar{V}_k$.

The reader will notice that, as the term V_k^2 appears in equation (37) only in association to buses with loads of the *constant-impedance* type, it is only needed to define μ_k and any other auxiliary variables or constraints for buses k in Ω_{ZCTE} .

The first of alternative for approximating V_k^2 requires that the auxiliary variable μ_k is bounded within a *convex envelope* for the V_k^2 . Taking into account that V_k is bounded within $\underline{V}_k \leq V_k \leq \bar{V}_k$, the *tightest* possible convex envelope for V_k^2 is defined with help of the following constraints:

$$\mu_k \geq (2 \cdot \underline{V}_k) \cdot V_k - \underline{V}_k^2, \forall k \in \Omega_{ZCTE} \quad (216)$$

$$\mu_k \geq (2 \cdot \bar{V}_k) \cdot V_k - \bar{V}_k^2, \forall k \in \Omega_{ZCTE} \quad (217)$$

$$\mu_k \leq (\underline{V}_k + \bar{V}_k) \cdot V_k - \underline{V}_k \cdot \bar{V}_k, \forall k \in \Omega_{ZCTE} \quad (218)$$

where:

μ_k Auxiliary variable that represents approximation of V_k^2 , for k in Ω_{ZCTE} .

The second alternative would be to build an approximation of the function V_k^2 , around a reference value V_k^0 , with basis on the corresponding Taylor series, truncated at the term of order 1. This would result in the following approximation:

$$\mu_k = (V_k^0)^2 + 2 \cdot V_k^0 \cdot (V_k - V_k^0), \forall k \in \Omega_{ZCTE} \quad (219)$$

where:

V_k^0 Reference voltage magnitude around which the approximation of V_k^2 based on a truncated Taylor series is constructed.

The reference value V_k^0 may be selected within $\underline{V}_k \leq V_k^0 \leq \bar{V}_k$ according to specific requirements of each application, keeping in mind that the quality of the approximation decreases as the distance among V_k and V_k^0 increases. Due to the fact that

voltages at the distribution system are (ideally) kept close to $V_k = 1$ p.u., choosing $V_k^0 = 1$ (and thus obtaining $\mu_k = 2 \cdot V_k - 1$) may be a reasonable choice (but not the only one) for several practical applications.

The two alternatives presented so far do not allow the user to arbitrate the accuracy of the approximation over the entire domain of the function V_k^2 (i.e., over the entire interval $\underline{V}_k \leq V_k \leq \bar{V}_k$). A third alternative for approximating V_k^2 , which allows achieving an arbitrary accuracy, is to employ a piecewise-linear approximation using SOS2, such as that described in section 3.2 of this dissertation.

In order to do that, it is possible to treat V_k^2 as a non-linear function of a single continuous decision variable, V_k , and to employ equations (48) to (53) to obtain the corresponding piecewise-linear approximation. This would, however, demand the definition of additional integer decision variables.

Alternatively, a piecewise-linear approximation that demands no additional integer decision variables can be constructed taking into account that V_k is a function of V_k^{re} and V_k^{im} . Thus, $V_k^2 = f[V_k(V_k^{re}, V_k^{im})] = g(V_k^{re}, V_k^{im}) = V_k^{re2} + V_k^{im2}$.

Taking into account that equations (78) to (85) have already been defined for all buses in $\{\Omega_B \setminus \Omega_{REF}\}$ (see section 4.2.1.2.1) and that $\Omega_{ZCTE} \subset \{\Omega_B \setminus \Omega_{REF}\}$, it becomes clear that it is only necessary to define the *function row* for obtaining a piecewise-linear approximation of $V_k^2 = V_k^{re2} + V_k^{im2}$. All other relevant constraints (the *reference row*, the *convexity row*, and the set of constraints that ensure that the weights $\lambda_k^{r,s}$ form a SOS2) have already been defined for buses in Ω_{ZCTE} .

Thus, the third alternative for approximating V_k^2 , which involves a piecewise-linear approximation of this function, requires only the definition of the following set of constraints:

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \lambda_k^{r,s} \cdot \hat{\mu}_k^{r,s} = \mu_k \quad , \forall k \in \Omega_{ZCTE} \quad (220)$$

where:

$\hat{\mu}_k^{r,s}$ Evaluated values of function V_k^2 , calculated at evaluation points $(\hat{V}_k^{re,r}, \hat{V}_k^{im,s})$ for bus k – i.e., $\hat{\mu}_k^{r,s} = \hat{V}_k^{re,r2} + \hat{V}_k^{im,s2}$.

4.2.2.5.2 Reformulation of $V_k \cdot (1 - \rho_k)$ and $V_k^2 \cdot (1 - \rho_k)$

Approximating $\mu_k \approx V_k^2$ by applying one of the techniques described in the previous subsection is only the first step to obtaining a reformulation of equation (37) that may be employed as the objective function of a mixed-integer linear program. It is also necessary to eliminate the products of decision variables $V_k \cdot (1 - \rho_k)$ and $V_k^2 \cdot (1 - \rho_k)$ – or, better said, $V_k \cdot (1 - \rho_k)$ and $\mu_k \cdot (1 - \rho_k)$.

Due to ρ_k being a binary decision variable, it is not necessary to construct *approximations* of the products $V_k \cdot (1 - \rho_k)$ and $\mu_k \cdot (1 - \rho_k)$. By introducing auxiliary decision variables and using disjunctive constraints, the exact values of these products can be represented in the objective function.

The product $V_k \cdot (1 - \rho_k)$ will be dealt with first. Every occurrence of it in the objective function is substituted by the auxiliary continuous decision variable γ_k^{ICTE} , and the following disjunctive constraints are defined:

$$M_k^{IOFl1} \cdot (1 - \rho_k) \leq \gamma_k^{ICTE} \leq M_k^{IOFu1} \cdot (1 - \rho_k), \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (221)$$

$$M_k^{IOFl2} \cdot \rho_k \leq \gamma_k^{ICTE} - V_k \leq M_k^{IOFu2} \cdot \rho_k, \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (222)$$

where:

γ_k^{ICTE} Auxiliary, continuous decision variable for modeling the product $V_k \cdot (1 - \rho_k)$, defined for buses k in $\{\Omega_{ICTE} \cap \Omega_{SHED}\}$;

M_k^{IOFl1} ; M_k^{IOFu1} ; M_k^{IOFl2} ; M_k^{IOFu2}

Disjunctive constants for disjunctive constraints for product $V_k \cdot (1 - \rho_k)$.

The definition of the constants M_k^{IOFl1} , M_k^{IOFu1} , M_k^{IOFl2} and M_k^{IOFu2} will be dealt with in section 4.3.1.

Analogously to what has been done for $V_k \cdot (1 - \rho_k)$, every occurrence of the product $\mu_k \cdot (1 - \rho_k)$ in the objective function should be replaced by the auxiliary continuous decision variable γ_k^{ZCTE} . The following disjunctive constraints are then defined:

$$M_k^{ZOF11} \cdot (1 - \rho_k) \leq \gamma_k^{ZCTE} \leq M_k^{ZOFu1} \cdot (1 - \rho_k) \quad , \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (223)$$

$$M_k^{ZOF12} \cdot \rho_k \leq \gamma_k^{ZCTE} - \mu_k \leq M_k^{ZOFu2} \cdot \rho_k \quad , \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (224)$$

where:

γ_k^{ZCTE} Auxiliary continuous decision variable for modeling the product $\mu_k \cdot (1 - \rho_k)$, defined for buses k in $\{\Omega_{ZCTE} \cap \Omega_{SHED}\}$;

M_k^{ZOF11} ; M_k^{ZOFu1} ; M_k^{ZOF12} ; M_k^{ZOFu2}

Disjunctive constants for disjunctive constraints for product $\mu_k \cdot (1 - \rho_k)$.

The definition of the constants M_k^{ZOF11} , M_k^{ZOFu1} , M_k^{ZOF12} and M_k^{ZOFu2} will be dealt with in section 4.3.1. The reader will notice that, as the associated disjunctive constraints are used for modeling the product $\mu_k \cdot (1 - \rho_k)$, the value of the disjunctive constants will depend on the approximation method employed to obtain μ_k .

4.2.2.5.3 Resulting objective function

After employing the approximation techniques listed in subsections 4.2.2.5.1 and 4.2.2.5.2 for treating the non-linear terms V_k^2 , $V_k^2 \cdot (1 - \rho_k)$ and $V_k \cdot (1 - \rho_k)$, the following reformulation of the objective function for the minimization of the cost of ohmic losses is obtained:

$$\begin{aligned} z^{LOSS} = \min \{ & c_k^{LOSS} \cdot \{ \sum_{k \in \Omega_{ITFC}} V_k^{ref} \cdot I_{g,k}^{re} + \sum_{k \in \Omega_{CTRPQ}} g_k^P \\ & + \sum_{k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\}} g_k^P + \sum_{k \in \Omega_{CURT}} g_k^P \cdot (1 - \tau_k) \\ & - [\sum_{k \in \{\Omega_{PCTE} \cap \Omega_{NSHD}\}} d_k^P + \sum_{k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\}} d_k^P \cdot (1 - \rho_k)] \\ & - [\sum_{k \in \{\Omega_{ICTE} \cap \Omega_{NSHD}\}} V_k \cdot d_k^P + \sum_{k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\}} \gamma_k^{ICTE} \cdot d_k^P] \\ & - \left[\sum_{k \in \{\Omega_{ZCTE} \cap \Omega_{NSHD}\}} \mu_k \cdot \frac{R_k^l}{|z_k^l|^2} + \sum_{k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\}} \gamma_k^{ZCTE} \cdot \frac{R_k^l}{|z_k^l|^2} \right] \} \} \quad (225) \end{aligned}$$

4.2.2.6 *Minimization of costs of reinforcements to the distribution system*

The following objective function, first introduced in section 2.2.2.6 and reproduced below for the sake of clarity, can be directly integrated to the MILP reformulation of the ACOPF:

$$z^{REIN} = \min\{\sum_{km \in \Psi_{CD}} c_{km}^{CONST} \cdot \sigma_{km}\} \quad (226)$$

4.2.2.7 *Minimization of costs of capacitor placement*

The following objective function, first introduced in section 2.2.2.7 and reproduced below for the sake of clarity, can also be directly integrated to the MILP reformulation of the ACOPF:

$$z^{CAPL} = \min\{\sum_{k \in \Omega_{CAP}} c_k^{CAPL} \cdot (1 - \rho_k)\} \quad (227)$$

4.2.2.8 *Minimization of circuit switching costs*

As well as in the two previous subsections, the following equation can be directly integrated to the MILP reformulation of the ACOPF. This objective function was first introduced in section 2.2.2.8 and is reproduced below for the sake of clarity.

$$z = \min\left\{\sum_{km \in \Psi_{SW}^{ON}} c_{km}^{SWITCH} \cdot (1 - \sigma_{km}) + \sum_{km \in \Psi_{SW}^{OFF}} c_{km}^{SWITCH} \cdot \sigma_{km}\right\} \quad (228)$$

4.3 Definition of parameters for linearization/convexification constraints

This section is dedicated to the calculation of the parameters necessary for employing linearization and convexification techniques to reformulate the ACOPF in distribution systems as a mixed-integer linear program:

- The definition of the *disjunctive constants* necessary for the definition of *disjunctive constraints* will be the object of section 4.3.1;
- The *evaluation points* and *evaluated values* necessary for the definition of *piecewise-linear approximations with SOS2* will be dealt with in section 4.3.2;
- The *upper and lower bounds* for the continuous variables whose product is modeled via *McCormick's envelope* will be the object of section 4.3.3.

It is important to explore the particular characteristics of the distribution system mentioned in section 2.1 (particularly, those of subsection 2.1.2), in order to be able to define numerical values for the abovementioned parameters that allow conciliating approximation accuracy and computational performance.

As will be seen in following subsections, many of the parameters of interest will be defined as a function of quantities related to bus voltages, particularly their real and imaginary components. Therefore, before moving on to subsections 4.3.1, 4.3.2 and 4.3.3, it is worth dedicating a few paragraphs to understand the intervals within which these real (V_k^{re}) and imaginary (V_k^{im}) components may vary – i.e., to characterize the domain of functions of the type $f(V_k^{re}, V_k^{im})$.

For that, it is first necessary to define an interval within which it is certain that each of the voltage angles within a typical distribution network may vary. This interval is denoted as $\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k, \forall k \in \Omega_B$.

In section 2.1.2, it has been stated that the voltage angles of all buses in a typical distribution network vary within a narrow interval around zero (provided that the reference angle of the reference bus, considered to be within the distribution network or right at its interface with the transmission system, its set to zero). The physical reasoning behind this statement has been presented in section 2.1.2 and will not be repeated here. The adjective *narrow*, however, does not correspond to a mathematical definition. A more precise definition would be to say that typical bus voltage angles within the distribution system vary in intervals such as $-5^\circ \leq \theta_k \leq 5^\circ$. While the exact lower and upper bounds vary from system to system, it is safe to say that, due to the characteristics mentioned in section 2.1.2, $|\underline{\theta}_k| \ll 90^\circ$ and $|\bar{\theta}_k| \ll 90^\circ$ for all practical distribution systems – this assumption will be considered for all further definitions and

mathematical manipulations in this dissertation. Actually, given that the reference angle has been set to zero and that a reference bus was chosen within the distribution system or at the interface with the transmission network, defining $-90^\circ \leq \theta_k \leq 90^\circ$ for any bus of a distribution network would be too conservative, and in practical distributions networks intervals such as $-5^\circ \leq \theta_k \leq 5^\circ$ would already suffice to capture the range of variation of the angles. The reader will notice that, for all case studies taken from the technical literature and simulated in chapter 5, using $-5^\circ \leq \theta_k \leq 5^\circ$ has been more than sufficient to capture the interval of variation of the bus voltage angles. It is important to emphasize that, for the discussion of this chapter, it is not required that $|\underline{\theta}_k| = |\bar{\theta}_k|$. For all further discussion, it is assumed, however, that $\underline{\theta}_k < 0$ and $\bar{\theta}_k > 0$.

Keeping this in mind, and recalling that bus voltage magnitudes are kept within the interval $\underline{V}_k \leq V_k \leq \bar{V}_k$ (typically, with \underline{V}_k near 0.95 p.u. and \bar{V}_k near 1.05 p.u.), it is possible to characterize the domain for functions of the type $f(V_k^{re}, V_k^{im})$.

First, the maximum and minimum values that V_k^{re} and V_k^{im} may assume can be calculated as:

$$V_k^{re,max} = \bar{V}_k \quad , \forall k \in \Omega_B \quad (229)$$

$$V_k^{re,min} = \underline{V}_k \cdot \cos[\max(|\underline{\theta}_k|, |\bar{\theta}_k|)] \quad , \forall k \in \Omega_B \quad (230)$$

$$V_k^{im,max} = \bar{V}_k \cdot \sin(\bar{\theta}_k) \quad , \forall k \in \Omega_B \quad (231)$$

$$V_k^{im,min} = \bar{V}_k \cdot \sin(\underline{\theta}_k) \quad , \forall k \in \Omega_B \quad (232)$$

where:

$$V_k^{re,min} ; V_k^{re,max}$$

Minimum and maximum values that V_k^{re} may assume;

$$V_k^{im,min} ; V_k^{im,max}$$

Minimum and maximum values that V_k^{im} may assume;

$\underline{\theta}_k ; \bar{\theta}_k$ Lower and upper bounds for the voltage angle at bus k (defined as inputs).

Above, reference is made to the definition of the bounds $\underline{\theta}_k$ and $\bar{\theta}_k$. As suggested by the subindex k , different bounds may be defined for each bus, if this is justified or allowed by some previous knowledge the user has on the distribution system

to be analyzed. If this knowledge is not available or if for any other reason defining different bounds for different buses is not desired, $\underline{\theta}_k = \underline{\theta}$ and $\bar{\theta}_k = \bar{\theta}$ may be used indistinctly for all buses. In fact, as the results of chapter 5 will show, for all simulated case studies considered in this dissertation, it has been sufficient to define bounds as conservative as $-5^\circ \leq \theta_k \leq 5^\circ$ for all buses.

It is important to emphasize that the bounds $\underline{\theta}_k$ and $\bar{\theta}_k$ are not directly used for defining of constraints of the type $\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k$. Rather than that, these bounds are employed for the definition of input parameters such as disjunctive constraints, evaluation points and evaluated values for piecewise-linear approximations, and extreme points for McCormick's envelopes.

The reader will notice that the superindices \blacksquare^{min} and \blacksquare^{max} , instead of the accents $\underline{\blacksquare}$ and $\bar{\blacksquare}$, have been used for characterizing the minimum and maximum values for V_k^{re} and V_k^{im} . As a general notation choice employed in this dissertation, $\underline{\blacksquare}$ and $\bar{\blacksquare}$ are used for characterizing bounds defined as inputs for the ACOPF, whereas \blacksquare^{min} and \blacksquare^{max} are used for calculating maximum and minimum values for variables with help of this input information.

Despite of the information on $V_k^{re,min}$, $V_k^{re,max}$, $V_k^{im,min}$ and $V_k^{im,max}$ being very useful (as will be seen in the next subsections), it does not provide as much insight on the domain of functions of the type $f(V_k^{re}, V_k^{im})$ as the reader may want at this point. A graphic characterization of the domain of these functions is shown in Figure 4.1, for different ranges of variation of θ_k and V_k .

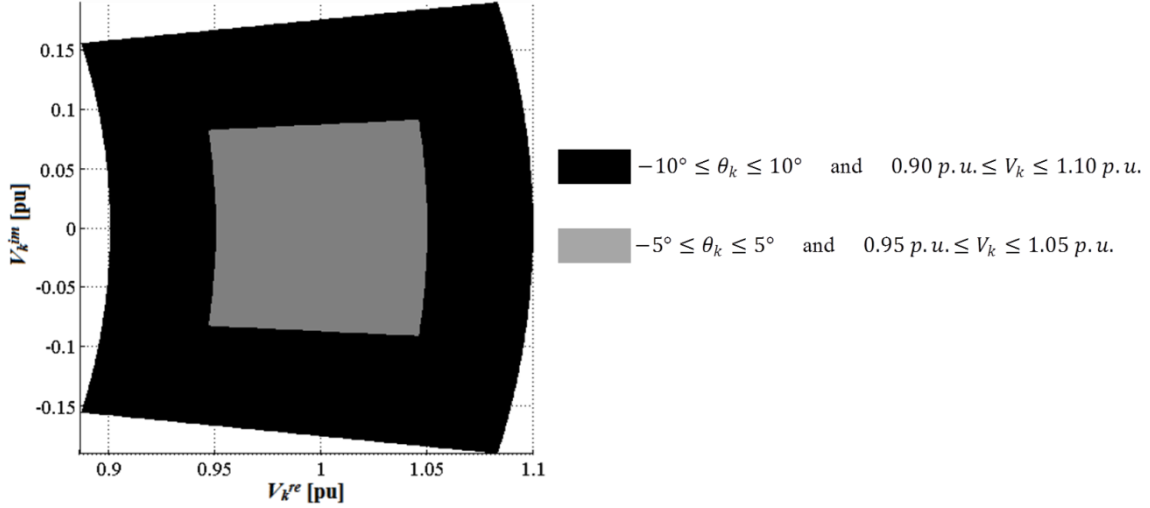


Figure 4.1: Domain for functions of the type $f(V_k^{re}, V_k^{im})$, for different ranges of variation of θ_k and V_k .

For Figure 4.1, the interval within which θ_k may vary is assumed to be symmetric about $\theta_k = 0$, and the interval within which V_k may vary is assumed to be symmetric about $V_k = 1$. However, these are not requirements for the concepts presented in this section.

Having provided the reader with some insight on the domain of functions of the type $f(V_k^{re}, V_k^{im})$, it is now time deal with the calculation of the parameters necessary to use linearization and convexification techniques for the reformulation of the ACOPF as a mixed-integer linear program.

4.3.1 Disjunctive constants

For the reasons exposed in section 3.1, defining *tight* disjunctive constants may affect the efficiency of solution techniques for mixed-integer programs. *Tight* constants are those that, while allowing the correct representation of disjunctions of the feasible space, have low numerical modulus (ideally, as low as possible). In this section, it will be shown how to define tight values for the disjunctive constraints used in the MILP.

4.3.1.1 Kirchhoff's laws for branches whose status can be altered

Constraints (73) and (74) of section 4.2.1.1 ensure that the real and imaginary components of the current flowing through an *inactive* branch are set to zero. Thus, the

value of the constant W_{km}^{CL} must be calculated so that, whenever the branch km is *active*, these constraints are relaxed. This can be done by defining:

$$W_{km}^{CL} = \bar{I}_{km} \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (233)$$

The definition above may be used independently of any analysis of the topology and loading conditions of the distribution system to be investigated. It is worth mentioning that, in theory, a *tighter* definition may be achieved by the solution of auxiliary optimization problems, with the objective of maximizing/minimizing the value of the current components while complying with a set of constraints that basically corresponds to that of the original problem. Nonetheless, the focus of this work is rather on the definition of disjunctive constants that may be readily obtained by simple manipulation of input parameters for the ACOPF – which corresponds to equation (233).

Equations (71) and (72), also from section 4.2.1.1, ensure that Kirchhoff's voltage law is relaxed whenever a branch km is *inactive*. Given that the real and imaginary components of the current flowing through that inactive branch will have been set to zero, it must be ensured that the disjunctive constraints are large enough to allow the free variation of the real and imaginary components of the terminal buses. This implicates in the following definition of the disjunctive constraints:

$$W_{km}^{VL, re, l} = (V_k^{re, min} - V_m^{re, max}) \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (234)$$

$$W_{km}^{VL, re, u} = (V_k^{re, max} - V_m^{re, min}) \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (235)$$

$$W_{km}^{VL, im, l} = (V_k^{im, min} - V_m^{im, max}) \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (236)$$

$$W_{km}^{VL, im, u} = (V_k^{im, max} - V_m^{im, min}) \quad , \forall km \in \{\Psi_{SW} \cup \Psi_{CD}\} \quad (237)$$

4.3.1.2 Loads

4.3.1.2.1 Constant-power loads that can be shed

Constraints (110) to (113) of section 4.2.1.3.2 ensure that the real and imaginary current components associated with a load that has been shed are set to null. The

definition of the disjunctive constants employed in these constraints will be dealt with in the following.

First, the following parameters are defined:

$$d_k^S = \sqrt{d_k^P{}^2 + d_k^Q{}^2} \quad , \forall k \in \Omega_{LOAD} \quad (238)$$

$$\phi_k = \text{atan}(d_k^Q/d_k^P) \quad , \forall k \in \Omega_{LOAD} \quad (239)$$

where:

- d_k^S Nominal apparent power demanded by load connected to bus k ;
- ϕ_k Apparent power angle (such that $d_k^P = d_k^S \cdot \cos\phi_k$ and $d_k^Q = d_k^S \cdot \sin\phi_k$) of load connected to bus k .

The reader will recall that the nominal values of the active and reactive power demanded by *constant-impedance* loads are $d_k^P = R_k^l/|Z_k^l|^2$ and $d_k^Q = X_k^l/|Z_k^l|^2$. Therefore, equations (238) and (239) apply to the calculation of the parameters d_k^S and ϕ_k for all types of loads.

These parameters will be employed in algebraic manipulations of the equations through which the real and imaginary components of loads of the *constant-power* type are obtained. Consider the following expressions:

$$I_{d,k}^{re} = (V_k^{re} \cdot d_k^P + V_k^{im} \cdot d_k^Q)/(V_k)^2 \quad , \forall k \in \Omega_{PCTE} \quad (240)$$

$$I_{d,k}^{im} = (V_k^{im} \cdot d_k^P - V_k^{re} \cdot d_k^Q)/(V_k)^2 \quad , \forall k \in \Omega_{PCTE} \quad (241)$$

These equations are obtained by substituting ξ_k and ζ_k with the equivalent expressions in terms of V_k^{re} and V_k^{im} . Equations (240) and (241) can be further manipulated, as indicated in the following.

The expression for $I_{d,k}^{re}$ will be dealt with first. For the following manipulation, the rectangular coordinates will be briefly abandoned, and polar coordinates will be employed:

$$I_{d,k}^{re} = (V_k \cdot \cos\theta_k \cdot d_k^S \cdot \cos\phi_k + V_k \cdot \sin\theta_k \cdot d_k^S \cdot \sin\phi_k) / (V_k)^2 \quad , \forall k \in \Omega_{PCTE} \quad (242)$$

$$I_{d,k}^{re} = \frac{V_k \cdot d_k^S}{(V_k)^2} \cdot (\cos\theta_k \cdot \cos\phi_k + \sin\theta_k \cdot \sin\phi_k) \quad , \forall k \in \Omega_{PCTE} \quad (243)$$

The expression inside parentheses obviously corresponds to $\cos(\theta_k - \phi_k)$, and (204) may be rewritten as:

$$I_{d,k}^{re} = \frac{d_k^S}{V_k} \cdot \cos(\theta_k - \phi_k) \quad , \forall k \in \Omega_{PCTE} \quad (244)$$

Keeping in mind that ϕ_k is a fixed parameter, the maximum and minimum values that $I_{d,k}^{re}$ may assume are given by:

$$I_{d,k}^{re,max} = d_k^S \cdot \max_{\substack{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \left\{ \frac{1}{V_k} \cdot \cos(\theta_k - \phi_k) \right\} \quad , \forall k \in \Omega_{PCTE} \quad (245)$$

$$I_{d,k}^{re,min} = d_k^S \cdot \min_{\substack{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \left\{ \frac{1}{V_k} \cdot \cos(\theta_k - \phi_k) \right\} \quad , \forall k \in \Omega_{PCTE} \quad (246)$$

where:

$$I_{d,k}^{re,min} ; I_{d,k}^{re,max}$$

Minimum and maximum values that $I_{d,k}^{re}$ may assume.

The evaluation of the expressions above is rather simple, but it is important to notice that, according to the values of $\underline{\theta}_k$, $\bar{\theta}_k$ and ϕ_k , $\cos(\theta_k - \phi_k)$ may assume negative values.

However, as mentioned in section 2.1.2, loads in the distribution system are incentivized to keep their power factor within narrow intervals. For instance, the Brazilian regulation prescribes incentives for the power factor of these loads to be bounded within $[0.92_{\text{lagging}}, 0.92_{\text{leading}}]$, resulting in ϕ_k bounded within $[-38.86^\circ, 38.86^\circ]$. With this range of typical values of ϕ_k , and considering that typical bus voltage angles within the distribution system vary in intervals such as $-5^\circ \leq \theta_k \leq 5^\circ$, $\cos(\theta_k - \phi_k)$ may only assume positive values. Even if the typical range of variation of the power

factor of distribution system loads were considered to be $[0.8_{\text{lagging}}, 0.8_{\text{leading}}]$ (a conservative assumption), the range of values for ϕ_k would be $[-59.20^\circ, 59.20^\circ]$, with the implication that $\cos(\theta_k - \phi_k)$ would still be able to assume only positive values for $-5^\circ \leq \theta_k \leq 5^\circ$.

Whenever the typical conditions mentioned in the last paragraph hold, resulting in $\cos(\theta_k - \phi_k)$ only assuming positive values, the expression for the calculation of the maximum and minimum values of the real current component may be rewritten as:

$$I_{d,k}^{re,max} = \frac{d_k^S}{V_k} \cdot \max_{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k} \{\cos(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{PCTE} \quad (247)$$

$$I_{d,k}^{re,min} = \frac{d_k^S}{V_k} \cdot \min_{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k} \{\cos(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{PCTE} \quad (248)$$

However, the reader should be careful in using equations (247) and (248) instead of (245) and (246). For some situations, using (247) and (248) may yield wrong values for $I_{d,k}^{re,max}$ and $I_{d,k}^{re,min}$ – e.g., when a purely capacitive load (which may be used for modeling a capacitor bank) is to be represented. This is due to the fact that a purely capacitive load has a power factor of zero. Obviously, for a purely capacitive load, the more general equations (245) and (246) must be used to calculate $I_{d,k}^{re,max}$ and $I_{d,k}^{re,min}$.

Now, the equivalent expressions for calculating $I_{d,k}^{im,max}$ and $I_{d,k}^{im,min}$ will be presented. Below, equation (241) is written in polar coordinates and manipulated:

$$I_{d,k}^{im} = (V_k \cdot \sin\theta_k \cdot d_k^S \cdot \cos\phi_k + V_k \cdot \cos\theta_k \cdot d_k^S \cdot \sin\phi_k) / (V_k)^2 \quad , \forall k \in \Omega_{PCTE} \quad (249)$$

$$I_{d,k}^{im} = \frac{V_k \cdot d_k^S}{(V_k)^2} \cdot (\sin\theta_k \cdot \cos\phi_k + \cos\theta_k \cdot \sin\phi_k) \quad , \forall k \in \Omega_{PCTE} \quad (250)$$

The expression inside parentheses obviously corresponds to $\sin(\theta_k - \phi_k)$, and (250) may be rewritten as:

$$I_{d,k}^{im} = \frac{d_k^S}{V_k} \cdot \sin(\theta_k - \phi_k) \quad , \forall k \in \Omega_{PCTE} \quad (251)$$

With ϕ_k fixed, the maximum and minimum values of $I_{d,k}^{im}$ are then calculated by:

$$I_{d,k}^{im,max} = d_k^S \cdot \max_{\substack{\theta_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \left\{ \frac{1}{V_k} \cdot \sin(\theta_k - \phi_k) \right\} \quad , \forall k \in \Omega_{PCTE} \quad (252)$$

$$I_{d,k}^{im,min} = d_k^S \cdot \min_{\substack{\theta_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \left\{ \frac{1}{V_k} \cdot \sin(\theta_k - \phi_k) \right\} \quad , \forall k \in \Omega_{PCTE} \quad (253)$$

where:

$$I_{d,k}^{im,min} ; I_{d,k}^{im,min}$$

Minimum and maximum values that $I_{d,k}^{im}$ may assume.

A further simplification of (252) and (253) for typical conditions is not possible, due to the fact that the sine function is an odd function. The following example illustrates the impossibility of simplification, even when the typical condition $-5^\circ \leq \theta_k \leq 5^\circ$ is considered: if $\phi_k = 38.86^\circ$, the expression for $I_{d,k}^{im,max}$ is given by $I_{d,k}^{im,max} = (d_k^S / \bar{V}_k) \cdot \sin(\bar{\theta}_k - \phi_k)$; yet, if $\phi_k = -38.86^\circ$, the expression for $I_{d,k}^{im,max}$ is $I_{d,k}^{im,max} = (d_k^S / \underline{V}_k) \cdot \sin(\bar{\theta}_k - \phi_k)$. The reader will notice that \bar{V}_k is the denominator in the first case, and \underline{V}_k is the denominator in the second case.

Having introduced the expressions for $I_{d,k}^{re,max}$, $I_{d,k}^{re,min}$, $I_{d,k}^{im,max}$ and $I_{d,k}^{im,min}$, the disjunctive constraints introduced in section 4.2.1.3.2 are now defined:

$$M_k^{D,re,PCTE,1} = -I_{d,k}^{re,max} \quad , \forall k \in \Omega_{PCTE} \quad (254)$$

$$M_k^{D,re,PCTE,2} = -I_{d,k}^{re,min} \quad , \forall k \in \Omega_{PCTE} \quad (255)$$

$$M_k^{D,re,PCTE,3} = I_{d,k}^{re,min} \quad , \forall k \in \Omega_{PCTE} \quad (256)$$

$$M_k^{D,re,PCTE,4} = I_{d,k}^{re,max} \quad , \forall k \in \Omega_{PCTE} \quad (257)$$

$$M_k^{D,im,PCTE,1} = -I_{d,k}^{im,max} \quad , \forall k \in \Omega_{PCTE} \quad (258)$$

$$M_k^{D,im,PCTE,2} = -I_{d,k}^{im,min} \quad , \forall k \in \Omega_{PCTE} \quad (259)$$

$$M_k^{D,im,PCTE,3} = I_{d,k}^{im,min} \quad , \forall k \in \Omega_{PCTE} \quad (260)$$

$$M_k^{D,im,PCTE,4} = I_{d,k}^{im,max} \quad , \forall k \in \Omega_{PCTE} \quad (261)$$

4.3.1.2.2 Constant-current loads that can be shed

Analogously to what has been done for *constant-power* loads, the first step to obtaining the values of the disjunctive constants for *constant-current* loads is rewriting the equations that relate the nominal value of the power demanded by the loads to the actual load currents, substituting η_k and κ_k by the expressions in terms of V_k^{re} and V_k^{im} :

$$I_{d,k}^{re} = (V_k^{re} \cdot d_k^p + V_k^{im} \cdot d_k^q) / V_k, \quad \forall k \in \Omega_{ICTE} \quad (262)$$

$$I_{d,k}^{im} = (V_k^{im} \cdot d_k^p - V_k^{re} \cdot d_k^q) / V_k, \quad \forall k \in \Omega_{ICTE} \quad (263)$$

In the following, the expression for $I_{d,k}^{re}$ is rewritten in polar coordinates and manipulated:

$$I_{d,k}^{re} = (V_k \cdot \cos\theta_k \cdot d_k^s \cdot \cos\phi_k + V_k \cdot \sin\theta_k \cdot d_k^s \cdot \sin\phi_k) / V_k, \quad \forall k \in \Omega_{ICTE} \quad (264)$$

$$I_{d,k}^{re} = \frac{V_k \cdot d_k^s}{V_k} \cdot (\cos\theta_k \cdot \cos\phi_k + \sin\theta_k \cdot \sin\phi_k), \quad \forall k \in \Omega_{ICTE} \quad (265)$$

$$I_{d,k}^{re} = d_k^s \cdot \cos(\theta_k - \phi_k), \quad \forall k \in \Omega_{ICTE} \quad (266)$$

Keeping in mind that ϕ_k is a fixed parameter, the maximum and minimum values that $I_{d,k}^{re}$ may assume are given by:

$$I_{d,k}^{re,max} = d_k^s \cdot \max_{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k} \{\cos(\theta_k - \phi_k)\}, \quad \forall k \in \Omega_{ICTE} \quad (267)$$

$$I_{d,k}^{re,min} = d_k^s \cdot \min_{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k} \{\cos(\theta_k - \phi_k)\}, \quad \forall k \in \Omega_{ICTE} \quad (268)$$

The reader will notice (267) and (268) are not functions of V_k . Therefore, no further simplification is required.

Now, the expression for $I_{d,k}^{im}$ is rewritten in polar coordinates and manipulated:

$$I_{d,k}^{im} = (V_k \cdot \sin\theta_k \cdot d_k^s \cdot \cos\phi_k + V_k \cdot \cos\theta_k \cdot d_k^s \cdot \sin\phi_k) / V_k, \quad \forall k \in \Omega_{ICTE} \quad (269)$$

$$I_{d,k}^{im} = \frac{V_k \cdot d_k^S}{V_k} \cdot (\sin\theta_k \cdot \cos\phi_k + \cos\theta_k \cdot \sin\phi_k) \quad , \forall k \in \Omega_{ICTE} \quad (270)$$

$$I_{d,k}^{im} = d_k^S \cdot \sin(\theta_k - \phi_k) \quad , \forall k \in \Omega_{ICTE} \quad (271)$$

With ϕ_k fixed, the maximum and minimum values of $I_{d,k}^{im}$ may be calculated by:

$$I_{d,k}^{im,max} = d_k^S \cdot \max_{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k} \{\sin(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ICTE} \quad (272)$$

$$I_{d,k}^{im,min} = d_k^S \cdot \min_{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k} \{\sin(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ICTE} \quad (273)$$

Having defined the expressions for $I_{d,k}^{re,max}$, $I_{d,k}^{re,min}$, $I_{d,k}^{im,max}$ and $I_{d,k}^{im,min}$, the disjunctive constraints introduced in section 4.2.1.3.4 may be defined:

$$M_k^{D,re,ICTE,1} = -I_{d,k}^{re,max} \quad , \forall k \in \Omega_{ICTE} \quad (274)$$

$$M_k^{D,re,ICTE,2} = -I_{d,k}^{re,min} \quad , \forall k \in \Omega_{ICTE} \quad (275)$$

$$M_k^{D,re,ICTE,3} = I_{d,k}^{re,min} \quad , \forall k \in \Omega_{ICTE} \quad (276)$$

$$M_k^{D,re,ICTE,4} = I_{d,k}^{re,max} \quad , \forall k \in \Omega_{ICTE} \quad (277)$$

$$M_k^{D,im,ICTE,1} = -I_{d,k}^{im,max} \quad , \forall k \in \Omega_{ICTE} \quad (278)$$

$$M_k^{D,im,ICTE,2} = -I_{d,k}^{im,min} \quad , \forall k \in \Omega_{ICTE} \quad (279)$$

$$M_k^{D,im,ICTE,3} = I_{d,k}^{im,min} \quad , \forall k \in \Omega_{ICTE} \quad (280)$$

$$M_k^{D,im,ICTE,4} = I_{d,k}^{im,max} \quad , \forall k \in \Omega_{ICTE} \quad (281)$$

4.3.1.2.3 Constant-impedance loads that can be shed

Keeping in mind that, for *constant-impedance* loads, the nominal values of the demanded active and reactive power are given by $d_k^P = R_k^l / |Z_k^l|^2$ and $d_k^Q = X_k^l / |Z_k^l|^2$, the expression that relates these nominal values to the actual load currents are given by:

$$I_{d,k}^{re} = (V_k^{re} \cdot d_k^P + V_k^{im} \cdot d_k^Q) \quad , \forall k \in \Omega_{ZCTE} \quad (282)$$

$$I_{d,k}^{im} = (V_k^{im} \cdot d_k^P - V_k^{re} \cdot d_k^Q) \quad , \forall k \in \Omega_{ZCTE} \quad (283)$$

The expression for $I_{d,k}^{re}$ may be rewritten in polar coordinates and manipulated:

$$I_{d,k}^{re} = (V_k \cdot \cos\theta_k \cdot d_k^S \cdot \cos\phi_k + V_k \cdot \sin\theta_k \cdot d_k^S \cdot \sin\phi_k) \quad , \forall k \in \Omega_{ZCTE} \quad (284)$$

$$I_{d,k}^{re} = V_k \cdot d_k^S \cdot (\cos\theta_k \cdot \cos\phi_k + \sin\theta_k \cdot \sin\phi_k) \quad , \forall k \in \Omega_{ZCTE} \quad (285)$$

$$I_{d,k}^{re} = V_k \cdot d_k^S \cdot \cos(\theta_k - \phi_k) \quad , \forall k \in \Omega_{ICTE} \quad (286)$$

Keeping in mind that ϕ_k is a fixed parameter, the maximum and minimum values that $I_{d,k}^{re}$ may assume are given by:

$$I_{d,k}^{re,max} = d_k^S \cdot \max_{\substack{\theta_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \{V_k \cdot \cos(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ZCTE} \quad (287)$$

$$I_{d,k}^{re,min} = d_k^S \cdot \min_{\substack{\theta_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \{V_k \cdot \cos(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ZCTE} \quad (288)$$

For loads to which typical conditions apply (i.e., $-59.20^\circ \leq \phi_k \leq 59.20^\circ$ and $-5^\circ \leq \theta_k \leq 5^\circ$, as discussed in section 4.3.1.2.1), $\cos(\theta_k - \phi_k)$ is only able to assume positive values. Therefore, whenever these typical conditions apply, the expression for the calculation of the maximum and minimum values of the real current component may be written as:

$$I_{d,k}^{re,max} = \bar{V}_k \cdot d_k^S \cdot \max_{\theta_k \leq \theta_k \leq \bar{\theta}_k} \{\cos(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ZCTE} \quad (289)$$

$$I_{d,k}^{re,min} = \underline{V}_k \cdot d_k^S \cdot \min_{\theta_k \leq \theta_k \leq \bar{\theta}_k} \{\cos(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ZCTE} \quad (290)$$

As for the case of *constant-power* loads, the reader should be careful while employing equations (289) and (290) instead of the more general forms (287) and (288). For instance, for a capacitor bank modeled as a purely capacity load, it is not possible to employ (289) and (290) – the more general expressions (287) and (288) must be used.

Now, the expression for $I_{d,k}^{im}$ is rewritten in polar coordinates and manipulated:

$$I_{d,k}^{im} = (V_k \cdot \sin\theta_k \cdot d_k^S \cdot \cos\phi_k + V_k \cdot \cos\theta_k \cdot d_k^S \cdot \sin\phi_k) \quad , \forall k \in \Omega_{ZCTE} \quad (291)$$

$$I_{d,k}^{im} = V_k \cdot d_k^S \cdot (\sin\theta_k \cdot \cos\phi_k + \cos\theta_k \cdot \sin\phi_k) \quad , \forall k \in \Omega_{ZCTE} \quad (292)$$

$$I_{d,k}^{im} = V_k \cdot d_k^S \cdot \sin(\theta_k - \phi_k) \quad , \forall k \in \Omega_{ZCTE} \quad (293)$$

With ϕ_k fixed, the maximum and minimum values of $I_{d,k}^{im}$ may be calculated by:

$$I_{d,k}^{im,max} = d_k^S \cdot \max_{\substack{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \{V_k \cdot \sin(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ZCTE} \quad (294)$$

$$I_{d,k}^{im,min} = d_k^S \cdot \min_{\substack{\underline{\theta}_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k}} \{V_k \cdot \sin(\theta_k - \phi_k)\} \quad , \forall k \in \Omega_{ZCTE} \quad (295)$$

Having defined the expressions for $I_{d,k}^{re,max}$, $I_{d,k}^{re,min}$, $I_{d,k}^{im,max}$ and $I_{d,k}^{im,min}$, the disjunctive constraints introduced in section 4.2.1.3.6 may be defined:

$$M_k^{D,re,ZCTE,1} = -I_{d,k}^{re,max} \quad , \forall k \in \Omega_{ZCTE} \quad (296)$$

$$M_k^{D,re,ZCTE,2} = -I_{d,k}^{re,min} \quad , \forall k \in \Omega_{ZCTE} \quad (297)$$

$$M_k^{D,re,ZCTE,3} = I_{d,k}^{re,min} \quad , \forall k \in \Omega_{ZCTE} \quad (298)$$

$$M_k^{D,re,ZCTE,4} = I_{d,k}^{re,max} \quad , \forall k \in \Omega_{ZCTE} \quad (299)$$

$$M_k^{D,im,ZCTE,1} = -I_{d,k}^{im,max} \quad , \forall k \in \Omega_{ZCTE} \quad (300)$$

$$M_k^{D,im,ZCTE,2} = -I_{d,k}^{im,min} \quad , \forall k \in \Omega_{ZCTE} \quad (301)$$

$$M_k^{D,im,ZCTE,3} = I_{d,k}^{im,min} \quad , \forall k \in \Omega_{ZCTE} \quad (302)$$

$$M_k^{D,im,ZCTE,4} = I_{d,k}^{im,max} \quad , \forall k \in \Omega_{ZCTE} \quad (303)$$

4.3.1.3 Generators

4.3.1.3.1 Curtailable generators with no control over the active power output

In section 4.2.1.2.2, disjunctive constraints have been introduced for modeling the curtailment of generators with no control over their active power output. This sections deals with the definition of the associated disjunctive constants.

The following are the (non-linear) expressions that relate the power output of generators to the associated current injections:

$$I_{g,k}^{re} = (V_k^{re} \cdot g_k^P + V_k^{im} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad , \forall k \in \Omega_{CTRQ} \quad (304)$$

$$I_{g,k}^{im} = (V_k^{im} \cdot g_k^P - V_k^{re} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad , \forall k \in \Omega_{CTRQ} \quad (305)$$

The reader will notice that the previous equations are structurally very similar to those relating the nominal power and currents of loads of the *constant-power* type. However, the fact that the reactive power output of generators, g_k^Q , is a decision variable that may vary in $\underline{g}_k^Q \leq g_k^Q \leq \overline{g}_k^Q$ introduces additional complexity in the calculation of the maximum and minimum values that the generation currents may assume. In fact, the method used in section 4.3.1.2, which involves expressing the power quantities in polar coordinates, would not facilitate the calculus of the maximum values that the current components may assume.

One technique that may be used for determining these values is to express only the voltage quantities in polar coordinates, thus obtaining:

$$I_{g,k}^{re} = (\cos\theta_k \cdot g_k^P + \sin\theta_k \cdot g_k^Q) / V_k \quad , \forall k \in \Omega_{CTRQ} \quad (306)$$

$$I_{g,k}^{im} = (\sin\theta_k \cdot g_k^P - \cos\theta_k \cdot g_k^Q) / V_k \quad , \forall k \in \Omega_{CTRQ} \quad (307)$$

The maximum and minimum values of the generation currents may be then obtained by solving the following equations:

$$I_{g,k}^{re,max} = \max_{\substack{\theta_k \leq \theta_k \leq \overline{\theta}_k \\ V_k \leq V_k \leq \overline{V}_k \\ \underline{g}_k^Q \leq g_k^Q \leq \overline{g}_k^Q}} \left\{ \frac{1}{V_k} \cdot (\cos\theta_k \cdot g_k^P + \sin\theta_k \cdot g_k^Q) \right\} \quad , \forall k \in \Omega_{CTRQ} \quad (308)$$

$$I_{g,k}^{re,min} = \min_{\substack{\theta_k \leq \theta_k \leq \overline{\theta}_k \\ V_k \leq V_k \leq \overline{V}_k \\ \underline{g}_k^Q \leq g_k^Q \leq \overline{g}_k^Q}} \left\{ \frac{1}{V_k} \cdot (\cos\theta_k \cdot g_k^P + \sin\theta_k \cdot g_k^Q) \right\} \quad , \forall k \in \Omega_{CTRQ} \quad (309)$$

$$I_{g,k}^{im,max} = \max_{\substack{\theta_k \leq \theta_k \leq \overline{\theta}_k \\ V_k \leq V_k \leq \overline{V}_k \\ \underline{g}_k^Q \leq g_k^Q \leq \overline{g}_k^Q}} \left\{ \frac{1}{V_k} \cdot (\sin\theta_k \cdot g_k^P - \cos\theta_k \cdot g_k^Q) \right\} \quad , \forall k \in \Omega_{CTRQ} \quad (310)$$

$$I_{g,k}^{im,min} = \min_{\substack{\theta_k \leq \theta_k \leq \bar{\theta}_k \\ \underline{V}_k \leq V_k \leq \bar{V}_k \\ \underline{g}_k^Q \leq g_k^Q \leq \bar{g}_k^Q}} \left\{ \frac{1}{V_k} \cdot (\sin\theta_k \cdot g_k^P - \cos\theta_k \cdot g_k^Q) \right\}, \forall k \in \Omega_{CTRQ} \quad (311)$$

where:

$$I_{g,k}^{re,min}; I_{g,k}^{re,max}; I_{g,k}^{im,min}; I_{g,k}^{im,max}$$

Minimum and maximum values that $I_{g,k}^{re}$ and $I_{g,k}^{im}$ may assume.

Solving the problems above, obtained after the transformation of the voltage quantities to polar coordinates, is slightly simpler than solving for the maxima and minima of $I_{g,k}^{re}$ and $I_{g,k}^{im}$ using equations (304) and (305), and considering additional constraints to ensure that $\underline{V}_k \leq V_k^{re2} + V_k^{im2} \leq \bar{V}_k$ and $\underline{V}_k \leq \text{atan}(V_k^{im}/V_k^{re}) \leq \bar{V}_k$.

However, if even the solution of (308) to (311) is deemed as problematic for any given reason, the user may resort to rough overestimators of $I_{g,k}^{re,max}$ and $I_{g,k}^{im,max}$ and rough underestimators of $I_{g,k}^{re,min}$ and $I_{g,k}^{im,min}$. A possible alternative for obtaining such rough underestimators and overestimators is to use the following expression:

$$I_{g,k}^{re,max} = I_{g,k}^{im,max} = -I_{g,k}^{re,min} = -I_{g,k}^{im,min} = \frac{1}{\underline{V}_k} \cdot \sqrt{g_k^{P2} + \left[\max(|\underline{g}_k^Q|, |\bar{g}_k^Q|) \right]^2}, \forall k \in \Omega_{CTRQ} \quad (312)$$

After obtaining $I_{g,k}^{re,max}$, $I_{g,k}^{im,max}$, $I_{g,k}^{re,min}$ and $I_{g,k}^{im,min}$ for all generators in Ω_{CTRQ} , the disjunctive constraints introduced in section 4.2.1.2.2 may be defined:

$$M_k^{G,re,1} = -I_{g,k}^{re,max}, \forall k \in \Omega_{CTRQ} \quad (313)$$

$$M_k^{G,re,2} = -I_{g,k}^{re,min}, \forall k \in \Omega_{CTRQ} \quad (314)$$

$$M_k^{G,re,3} = I_{g,k}^{re,max}, \forall k \in \Omega_{CTRQ} \quad (315)$$

$$M_k^{G,re,4} = I_{g,k}^{re,min}, \forall k \in \Omega_{CTRQ} \quad (316)$$

$$M_k^{G,im,1} = -I_{g,k}^{im,max}, \forall k \in \Omega_{CTRQ} \quad (317)$$

$$M_k^{G,im,2} = -I_{g,k}^{im,min}, \forall k \in \Omega_{CTRQ} \quad (318)$$

$$M_k^{G,im,3} = I_{g,k}^{im,min} \quad , \forall k \in \Omega_{CTRQ} \quad (319)$$

$$M_k^{G,im,4} = I_{g,k}^{im,max} \quad , \forall k \in \Omega_{CTRQ} \quad (320)$$

4.3.1.4 Terms of the objective function for minimization of losses

A number of disjunctive constraints have been defined in section 4.2.2.5.2 for the reformulation of the products $V_k \cdot (1 - \rho_k)$ and $V_k^2 \cdot (1 - \rho_k)$. Those will be dealt with in the following subsections.

4.3.1.4.1 Reformulation of the product $V_k \cdot (1 - \rho_k)$

As indicated below, the value of the disjunctive constants employed in the reformulation of the product $V_k \cdot (1 - \rho_k)$ can be determined exclusively with basis on inputs for the ACOPF – namely the bounds for voltage magnitudes of the buses in $\Omega_{ICTE} \cap \Omega_{SHED}$.

$$M_k^{IOFl1} = \underline{V}_k \quad , \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (321)$$

$$M_k^{IOFu1} = \overline{V}_k \quad , \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (322)$$

$$M_k^{IOFl2} = -\overline{V}_k \quad , \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (323)$$

$$M_k^{IOFu2} = -\underline{V}_k \quad , \forall k \in \{\Omega_{ICTE} \cap \Omega_{SHED}\} \quad (324)$$

4.3.1.4.2 Reformulation of the product $\mu_k \cdot (1 - \rho_k)$

As indicated in section 4.2.2.5.1, μ_k represents an approximation for the term V_k^2 . As the disjunctive constants M_k^{ZOF11} , M_k^{ZOFu1} , M_k^{ZOF12} and M_k^{ZOFu2} are used in the reformulation of the product $\mu_k \cdot (1 - \rho_k)$, the definition of these constants will depend on the approximation method used to obtain μ_k .

In the following, three different definitions of the disjunctive constants are presented. Each of these is associated with one alternative method for approximating μ_k presented in section 4.2.2.5.1.

4.3.1.4.2.1 Approximation of μ_k via McCormick's envelope

If μ_k is approximated via a convex envelope, the following definition applies:

$$M_k^{ZOF11} = (\underline{V}_k)^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (325)$$

$$M_k^{ZOFu1} = (\overline{V}_k)^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (326)$$

$$M_k^{ZOF12} = -(\overline{V}_k)^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (327)$$

$$M_k^{ZOFu2} = -(\underline{V}_k)^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (328)$$

4.3.1.4.3 Approximation of μ_k via truncated Taylor series

If the approximation via truncated Taylor series has been used, the disjunctive constants should be defined as follows:

$$M_k^{ZOF11} = [(V_k^0)^2 + 2 \cdot V_k^0 \cdot (\underline{V}_k - V_k^0)]^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (329)$$

$$M_k^{ZOFu1} = [(V_k^0)^2 + 2 \cdot V_k^0 \cdot (\overline{V}_k - V_k^0)]^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (330)$$

$$M_k^{ZOF12} = -[(V_k^0)^2 + 2 \cdot V_k^0 \cdot (\overline{V}_k - V_k^0)]^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (331)$$

$$M_k^{ZOFu2} = -[(V_k^0)^2 + 2 \cdot V_k^0 \cdot (\underline{V}_k - V_k^0)]^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (332)$$

4.3.1.4.4 Term μ_k obtained via piecewise-linear approximation with SOS2

In this case, the disjunctive constraints should be defined as shown below:

$$M_k^{ZOF11} = \{\max_{r \in \Gamma^{re}, s \in \Gamma^{im}} (\hat{V}_k^{re, r^2} + \hat{V}_k^{im, s^2})\}^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (333)$$

$$M_k^{ZOFu1} = \{\min_{r \in \Gamma^{re}, s \in \Gamma^{im}} (\hat{V}_k^{re, r^2} + \hat{V}_k^{im, s^2})\}^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (334)$$

$$M_k^{ZOF12} = -\{\max_{r \in \Gamma^{re}, s \in \Gamma^{im}} (\hat{V}_k^{re, r^2} + \hat{V}_k^{im, s^2})\}^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (335)$$

$$M_k^{ZOFu2} = -\{\min_{r \in \Gamma^{re}, s \in \Gamma^{im}} (\hat{V}_k^{re, r^2} + \hat{V}_k^{im, s^2})\}^2, \quad \forall k \in \{\Omega_{ZCTE} \cap \Omega_{SHED}\} \quad (336)$$

4.3.2 Evaluation points for piecewise-linearization with SOS2

As indicated in section 3.2, there is a trade-off between approximation accuracy and computational performance while defining the number of evaluation points for piecewise-linear approximations of non-convex, non-linear functions, with help of SOS2.

In this section, we present the choice of the number and location of these evaluation points which has been used in this dissertation. The set of evaluation points (and consequently evaluated values) presented here, while not necessarily optimal, led to the accuracy and computational performance results shown in chapter 5, which are deemed as satisfactory for an initial investigation.

There are a number of reasons for which the procedure presented here is not optimal. The first is that, as indicated in section 3.2, the choice of the evaluation points is based on rectangular partitions of the domain of the non-linear, non-convex functions, whereas there is evidence (see [61], [80]) that *triangular* partitions may lead to better computational performance. Furthermore, the set of evaluation points presented here has been chosen basically via a trial-and-error procedure, guided by knowledge of the ACOPF problem and of the shape of the functions to be approximated. Particular emphasis has been given to defining a set of evaluation points whose convex hull includes the whole domain of the functions of two arguments – i.e., that any point in the domain could be achieved by affine combination of evaluation points. But yet, no techniques that ensure that the choice of points is optimal either with respect to accuracy (e.g., minimizing the maximum approximation error while keeping the number of points below a certain threshold) or computational performance (e.g., minimizing the number of points while keeping the maximum approximation error below a certain threshold) have been used. The investigation of such techniques is listed among possible topics for future work.

Now that the reader has been warned of the potentially sub-optimal character of the procedure employed for the definition of evaluation points, the set of points chosen for each piecewise-linear approximation introduced in section 4.2 will be presented in the following subsections.

4.3.2.1 Evaluation points for functions of V_k^{re} and V_k^{im}

Various functions that have V_k^{re} and V_k^{im} as arguments have been introduced in the previous sections: ξ_k and ζ_k in sections 4.2.1.2 and 4.2.1.3; η_k and κ_k in sections 4.2.1.3.3 and 4.2.1.3.4; V_k in section 4.2.1.4.1; and V_k^2 in section 4.2.2.5.1.

The definition of the evaluation points for these functions may take advantage of the fact that, due to physical characteristics of the distribution system (low R/X ratio, voltage magnitude kept within narrow limits, etc.), the domain of the functions of interest in the Cartesian coordinate system (V_k^{re}, V_k^{im}) has the particular shape indicated in Figure 4.1. Among the most important characteristics of this domain is that *it does not include* the point $(V_k^{re} = 0, V_k^{im} = 0)$. In fact, it excludes all points for which $\sqrt{(V_k^{re})^2 + (V_k^{im})^2} < \underline{V}_k$. This is an important feature, as some of the functions to be approximated have either the term $(V_k^{re})^2 + (V_k^{im})^2$ or its square root in their denominator, meaning that the approximation of the non-linearities would become increasingly more demanding as the point $(V_k^{re} = 0, V_k^{im} = 0)$ were approached.

The first alternative to determining the sets of evaluation points $\{\hat{V}_k^{re}\}$ and $\{\hat{V}_k^{im}\}$ would be to first define the cardinality of each set and then to distribute the correspondent number of points evenly within the intervals $[V_k^{re,min}, V_k^{re,max}]$ (for $\{\hat{V}_k^{re}\}$) and $[V_k^{im,min}, V_k^{im,max}]$ (for $\{\hat{V}_k^{im}\}$), making sure to include evaluation points corresponding to the extreme values of each interval. This approach is described in more detail in sections 4.3.2.2 and 4.3.2.3, in which functions of other decision variables are dealt with. It is worth mentioning that, as the interval $[V_k^{im,min}, V_k^{im,max}]$ includes negative and positive values, it is recommended that the point $\hat{V}_k^{im} = 0$ is included in $\{\hat{V}_k^{im}\}$, as described in detail in section 4.3.2.3.

However, a second possible definition of the evaluation points, which has been obtained empirically and has led to slightly better results than using equally-spaced points within the intervals of interest (with respect both to accuracy and computational performance, for sets of the same cardinality), will be presented below.

The set of evaluation points $\{\hat{V}_k^{re}\}$ corresponding to this second alternative, which has cardinality $|\{\hat{V}_k^{re}\}| = |\Gamma^{re}| = 5$, is that defined through equation (337):

$$\{\hat{V}_k^{re}\} = \left\{ \begin{array}{c} V_k^{re,min} \\ \underline{V}_k \\ \frac{1}{2} \cdot (\underline{V}_k + \bar{V}_k) \\ \bar{V}_k \cdot \cos[\max(|\underline{\theta}_k|, |\bar{\theta}_k|)] \\ V_k^{re,max} \end{array} \right\} \quad (337)$$

The characteristics of this set of evaluation points will be discussed further in this section, with help of graphic information. Before that, the set of evaluation points $\{\hat{V}_k^{im}\}$ corresponding to this second approach is presented. This set, whose cardinality is $|\{\hat{V}_k^{im}\}| = |\Gamma^{im}| = 9$, is defined by:

$$\{\hat{V}_k^{im}\} = \left\{ \begin{array}{c} V_k^{im,min} \\ \underline{V}_k \cdot \sin(\underline{\theta}_k) \\ \frac{1}{2} \cdot \underline{V}_k \cdot [\sin(\underline{\theta}_k) + \sin(\underline{\theta}_k/2)] \\ \underline{V}_k \cdot \sin(\underline{\theta}_k/2) \\ 0 \\ \underline{V}_k \cdot \sin(\bar{\theta}_k/2) \\ \frac{1}{2} \cdot \underline{V}_k \cdot [\sin(\bar{\theta}_k) + \sin(\bar{\theta}_k/2)] \\ \underline{V}_k \cdot \sin(\bar{\theta}_k) \\ V_k^{im,max} \end{array} \right\} \quad (338)$$

It is clear the sets defined above include the extreme values $V_k^{re,min}$, $V_k^{re,max}$, $V_k^{im,min}$ and $V_k^{im,max}$. This ensures that the convex hull of the set of points $\{\langle \hat{V}_k^{re}, \hat{V}_k^{im} \rangle\}$ includes the entire domain of (V_k^{re}, V_k^{im}) .

At this point, the reader's comprehension of the nature of piecewise-linear approximations of functions of two decision variables may be enhanced with the display of graphical information. For the following discussion, the definition of the evaluation points corresponding to equations (337) and (338) has been considered.

The reader is thus invited to first consider Figure 4.2, in which the domain (V_k^{re}, V_k^{im}) is indicated in black, while the set of evaluation points $\{\langle \hat{V}_k^{re}, \hat{V}_k^{im} \rangle\}$, obtained by the Cartesian product of the sets $\{\hat{V}_k^{re}\}$ and $\{\hat{V}_k^{im}\}$, is indicated by white

dots. The intervals $0.95 \leq V_k \leq 1.05$ and $-5^\circ \leq \theta_k \leq 5^\circ$ have been considered for this and all subsequent figures of section 4.3.2.1.

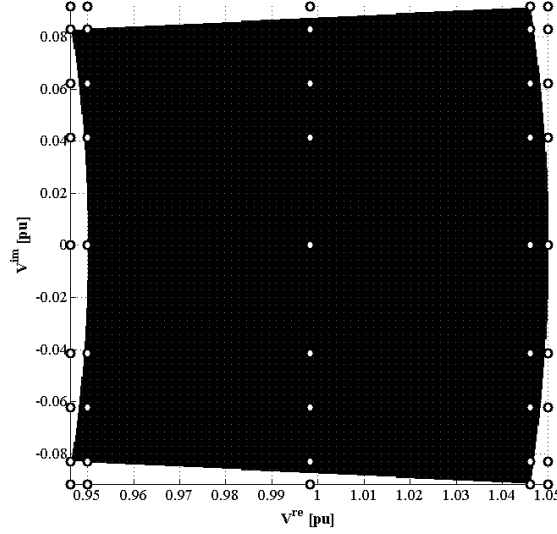


Figure 4.2: Domain (V_k^{re}, V_k^{im}) and set of evaluation points $\{\langle \hat{V}_k^{re}, \hat{V}_k^{im} \rangle\}$ obtained by the Cartesian product of the sets defined in equations (300) and (301).

It is clear that, while the convex hull of $\{\langle \hat{V}_k^{re}, \hat{V}_k^{im} \rangle\}$ includes the domain (V_k^{re}, V_k^{im}) , it does not coincide with it. This is not a problem from the point of view of the adequacy of the representation of the domain, as other constraints of the MILP formulation (e.g., $\underline{V}_k \leq V_k \leq \bar{V}_k$) will ensure that the (approximate) solution of the problem lies within the correct domain. However, the non-coincidence of the convex hull of $\{\langle \hat{V}_k^{re}, \hat{V}_k^{im} \rangle\}$ and the domain (V_k^{re}, V_k^{im}) points out to an inefficiency of the definition of the evaluation points – it is clear that a triangular partition of the feasible space would potentially reduce the required number of evaluation points.

The following figures allow the graphical evaluation both of the non-linear functions of (V_k^{re}, V_k^{im}) and of the corresponding piecewise-linear approximations. In each of the figures, the non-linear function is shown on the left side and the piecewise-linear approximation on the right, with white dots indicating the position of the evaluation points in both graphs. Due to limitations of the plotting procedures, both the original function and the piecewise-linear approximation are shown for the region defined by the convex hull of $\{\langle \hat{V}_k^{re}, \hat{V}_k^{im} \rangle\}$, and not for the original domain – the reader should thus keep in mind that other constraints of the MILP will only allow that points that lie within the original domain are visited.

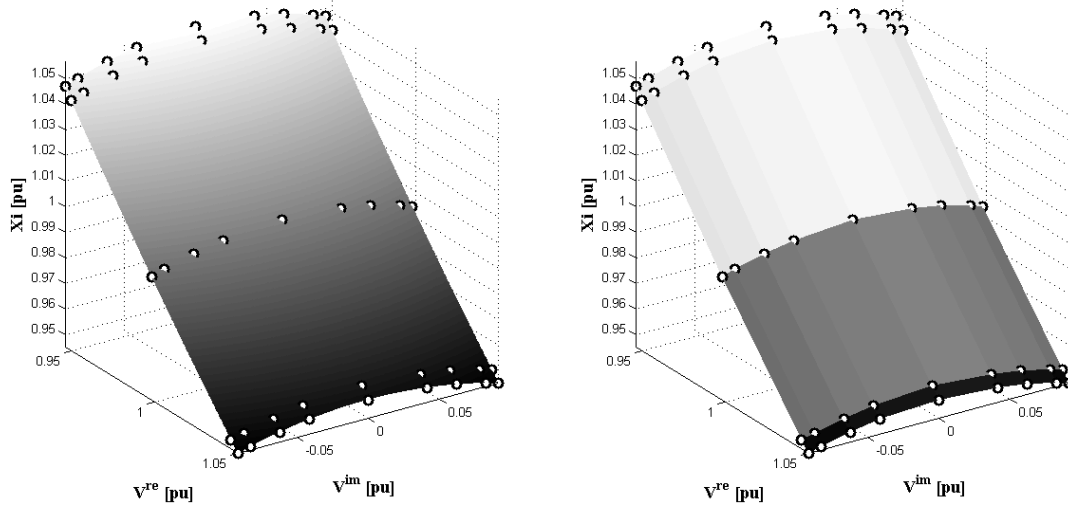


Figure 4.3: Depiction of ξ_k : non-linear function (left) and piecewise-linear approximation (right).

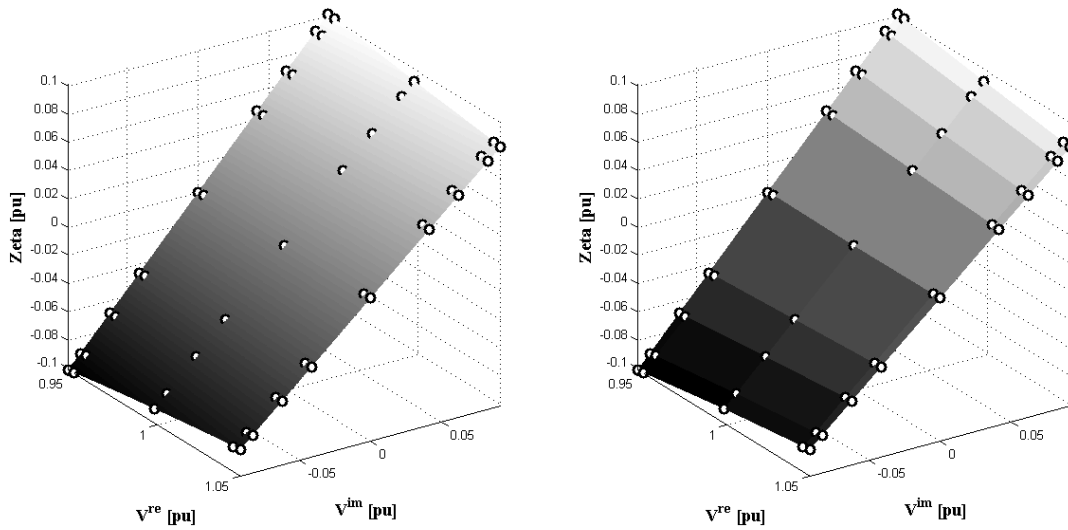


Figure 4.4: Depiction of ζ_k : non-linear function (left) and piecewise-linear approximation (right).

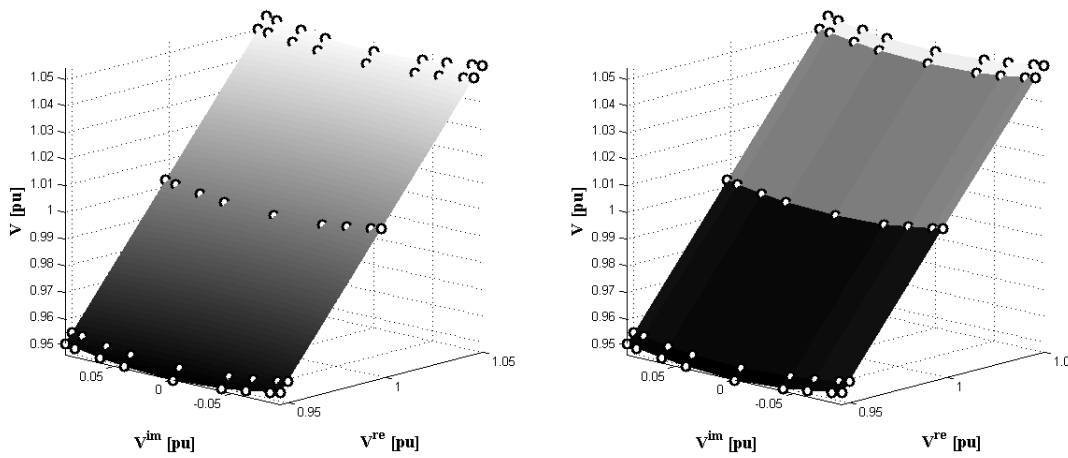


Figure 4.5: Depiction of V_k : non-linear function (left) and piecewise-linear approximation (right).

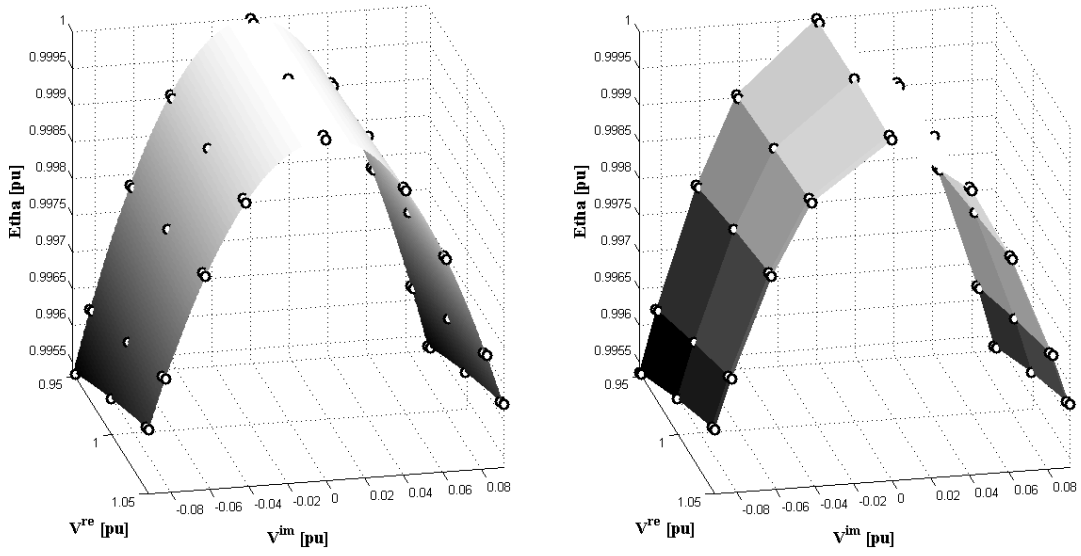


Figure 4.6: Depiction of η_k : non-linear function (left) and piecewise-linear approximation (right).

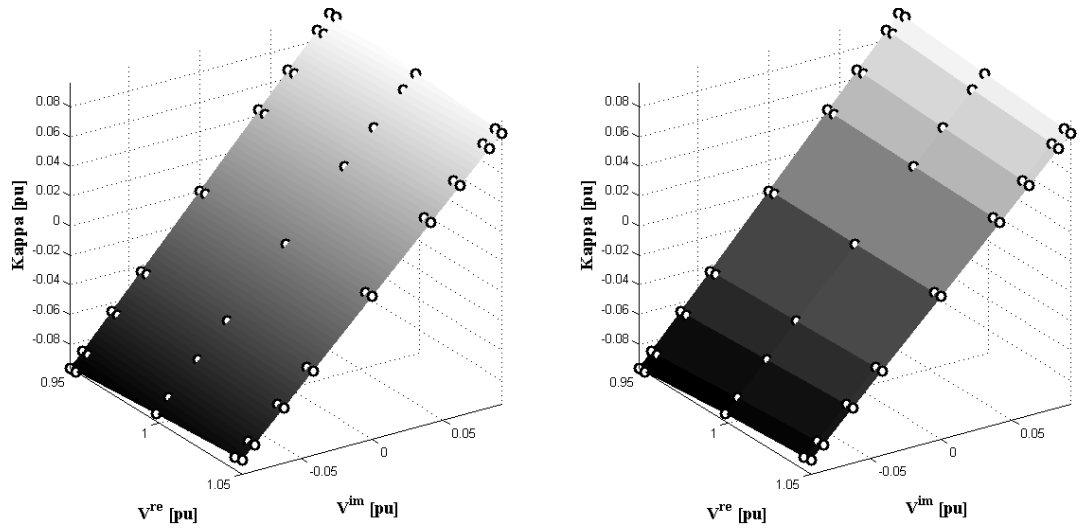


Figure 4.7: Depiction of κ_k : non-linear function (left) and piecewise-linear approximation (right).

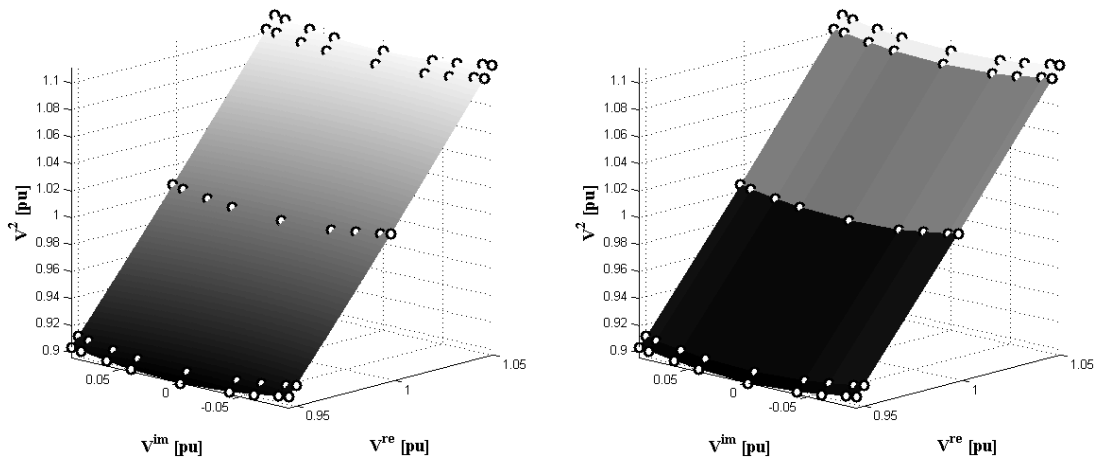


Figure 4.8: Depiction of V_k^2 : non-linear function (left) and piecewise-linear approximation (right).

4.3.2.2 Evaluation points for a function of ι_{km}^{re} and ι_{km}^{im}

As indicated in section 4.2.1.4.2, the auxiliary variables ι_{km}^{re} are ι_{km}^{im} are at least as high as the modulus of the branch current components I_{km}^{re} and I_{km}^{im} . Due to that, and to the fact that each of the current components may assume any value within the interval $[-\bar{I}_{km}, \bar{I}_{km}]$, ι_{km}^{re} are ι_{km}^{im} will vary in $0 \leq \iota_{km}^{re} \leq \bar{I}_{km}$ and $0 \leq \iota_{km}^{im} \leq \bar{I}_{km}$, respectively. However, the reader will notice that the constraint $\iota_{km} \leq \bar{I}_{km}$ limits the domain of

interest of the function $\iota_{km} = \sqrt{(\iota_{km}^{re})^2 + (\iota_{km}^{im})^2}$.

The set of evaluation points $\{\hat{\iota}_{km}^{re}\}$ and $\{\hat{\iota}_{km}^{im}\}$ used in this dissertation corresponds to an equally-spaced partition of the intervals $0 \leq \iota_{km}^{re} \leq \bar{I}_{km}$ and $0 \leq \iota_{km}^{im} \leq \bar{I}_{km}$.

Keeping in mind that $|\Pi^{re}| = |\{\hat{\iota}_{km}^{re}\}|$, $\{\hat{\iota}_{km}^{re}\}$ can be written as:

$$\{\hat{\iota}_{km}^{re}\} = \left\{ \bar{I}_{km} \cdot \frac{(r-1)}{(|\Pi^{re}|-1)} \mid r \in \Pi^{re} \right\} \quad (339)$$

Analogously, the set of evaluation points $\{\hat{\iota}_{km}^{im}\}$ may be written as:

$$\{\hat{\iota}_{km}^{im}\} = \left\{ \bar{I}_{km} \cdot \frac{(s-1)}{(|\Pi^{im}|-1)} \mid s \in |\Pi^{im}| \right\} \quad (340)$$

Figure 4.9¹¹ depicts the function ι_{km} and its piecewise-linear approximation, obtained for $|\Pi^{re}| = |\Pi^{im}| = 5$. The corresponding set of evaluation points $\{\langle \hat{\iota}_{km}^{re}, \hat{\iota}_{km}^{im} \rangle\}$ is indicated by white dots.

¹¹ In Figure 4.9, it is assumed that $\bar{I}_{km} = 1$ p.u. Depending on the apparent power basis considered, this would be an overly overestimated limit. Figure 4.9 aims merely at providing the reader with insight on the shape of the non-linear function and its approximation.

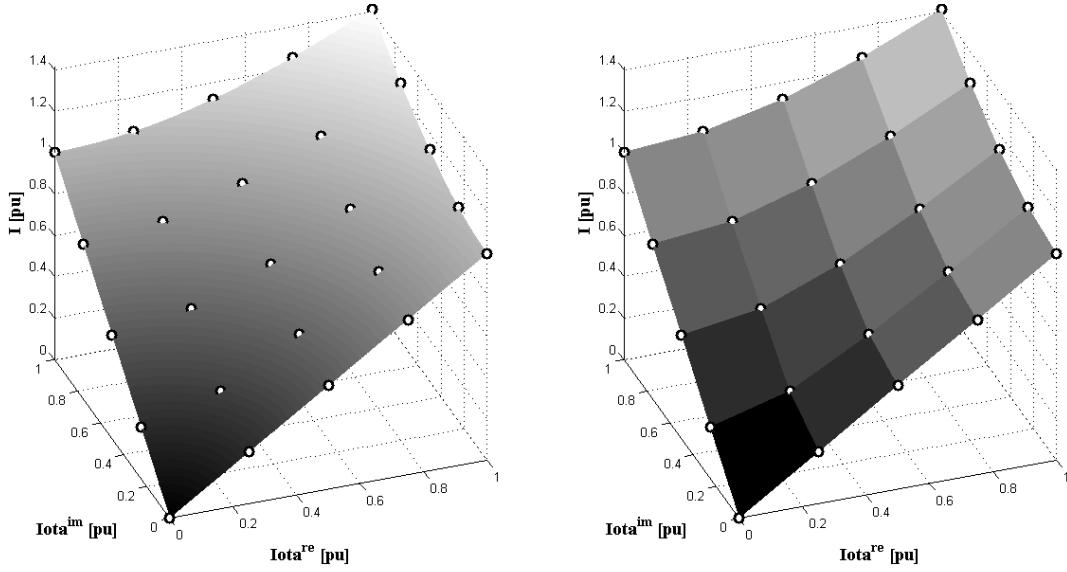


Figure 4.9: Depiction of ι_{km} : non-linear function (left) and piecewise-linear approximation (right) obtained with $|\Pi^{re}| = |\Pi^{im}| = 5$. The set of evaluation points $\{\{\hat{\iota}_{km}^{re}, \hat{\iota}_{km}^{im}\}\}$ is indicated by white dots.

Here, as well as in section 4.3.2.1, the non-coincidence of the convex hull of $\{\{\hat{\iota}_{km}^{re}, \hat{\iota}_{km}^{im}\}\}$ and the region limited by the constraint $\iota_{km} \leq \bar{\iota}_{km}$ points out to an inefficiency of the definition of the evaluation points – for instance, as the point $\{\{\hat{\iota}_{km}^{re} = \bar{\iota}_{km}, \hat{\iota}_{km}^{im} = \bar{\iota}_{km}\}\}$ will never be reached, moving it closer to the origin would presumably enhance the accuracy of the approximation.

4.3.2.3 Evaluation points for a function of V_k and $I_{g,k}^{re}$

In section 4.2.2.4.2, the construction of a piecewise-linear approximation of the bilinear product $V_k \cdot I_{g,k}^{re}$ has been suggested as one of the alternatives for incorporating it into a MILP.

In the following discussion, reference will be made to the bounds of the decision variables V_k and $I_{g,k}^{re}$. The bounds for V_k (\bar{V}_k and \underline{V}_k) are input parameters for the ACOPF, and need no further explanation.

Considering only the *physical* characteristics of the problem of minimization of imports, it would not be necessary to impose any bounds to the real component of the *slack current* $I_{g,k}^{re}$ – i.e., it may be in the interest of the user to assume that the bus at the interface with the transmission system is capable of meeting any power import demands (it is an *infinite bus*). However, due to the need to reformulate the product $V_k \cdot I_{g,k}^{re}$, it is necessary to define the limits of the interval within which $I_{g,k}^{re}$ may vary. One possible

way of doing that is to define, with basis on knowledge about the system under analysis, a (conservative) estimate of the minimum amount of power that will need to be imported through bus k (which may be negative if exports are also possible), \underline{p}_k , and an (conservative) estimate of the maximum amount of power to be exported through bus k , \bar{p}_k . As V_k is always non-negative, the bounds on $I_{g,k}^{re}$ would then be calculated as:

$$\underline{I}_{g,k}^{re} = \begin{cases} \underline{p}_k / \underline{V}_k & , \underline{p}_k < 0 \\ \underline{p}_k / \bar{V}_k & , \underline{p}_k \geq 0 \end{cases} \quad , \forall k \in \Omega_{ITFC} \quad (341)$$

$$\bar{I}_{g,k}^{re} = \begin{cases} \bar{p}_k / \bar{V}_k & , \bar{p}_k < 0 \\ \bar{p}_k / \underline{V}_k & , \bar{p}_k \geq 0 \end{cases} \quad , \forall k \in \Omega_{ITFC} \quad (342)$$

where:

$\underline{p}_k ; \bar{p}_k$ Estimates for minimum and maximum power to be imported through bus k .

The set of evaluation points $\{\hat{V}_k\}$ used in this dissertation corresponds to an equally-spaced partition of the interval $\underline{V}_k \leq V_k \leq \bar{V}_k$. Denoting the cardinality of the set E^V by $|E^V|$, the following expression may be used for defining $\{\hat{V}_k\}$:

$$\{\hat{V}_k\} = \left\{ \underline{V}_k + (\bar{V}_k - \underline{V}_k) \cdot \frac{(r-1)}{(|E^V|-1)} \mid r \in E^V \right\} \quad (343)$$

The definition of the set of evaluation points $\{\hat{I}_{g,k}^{re}\}$ is somewhat more complex. If $I_{g,k}^{re}$ assumes strictly non-negative values *or* strictly non-positive values, the following expression may be used:

$$\{\hat{I}_{g,k}^{re}\} = \left\{ \underline{I}_{g,k}^{re} + (\bar{I}_{g,k}^{re} - \underline{I}_{g,k}^{re}) \cdot \frac{(s-1)}{(|E^{Ig}|-1)} \mid r \in E^{Ig} \right\} \quad (344)$$

If $I_{g,k}^{re}$ may assume *both* negative *and* positive values, it is important to include the value zero within the set $\{\hat{I}_{g,k}^{re}\}$. One possible alternative for defining the set is then to use n^{neg} negative evaluation points and n^{pos} positive evaluation points, and define the set of evaluation points as:

$$\begin{aligned} \{\hat{I}_{g,k}^{re}\} = \\ \{I_{g,k}^{re} \cdot (1 - \frac{r-1}{n^{neg}}) \mid r \in \{1 \dots n^{neg}\}\} \cup \{0\} \cup \{\bar{I}_{g,k}^{re} \cdot \frac{r}{n^{pos}} \mid r \in \{1 \dots n^{pos}\}\} \end{aligned} \quad (345)$$

In this case, the cardinality of $\{\hat{I}_{g,k}^{re}\}$ is $n^{neg} + n^{pos} + 1$.

4.3.3 Bounds for continuous decision variables in bilinear products

As mentioned in section 3.3, the accuracy of the approximation of bilinear products via McCormick's envelope is dictated by how *tight* one is able to define the upper and lower bounds of the continuous variables that form the product. Ideally, the definition of the bounds should correspond to the tightest *de facto* interval within which the continuous variables may vary. In this section, it will be shown how to define tight values for the bounds of variables whose bilinear products are approximated via McCormick's envelope, for the proposed MILP reformulation of the ACOFP.

4.3.3.1 Bounds for ξ_k e ζ_k

The auxiliary variables ξ_k and ζ_k have appeared in the bilinear products $\xi_k \cdot g_k^Q$, $\zeta_k \cdot g_k^Q$, $\xi_k \cdot g_k^P$ and $\zeta_k \cdot g_k^P$ in section 4.2.1.2. The bounds for the reactive and active output of the generators are inputs for the ACOFP, but it is still needed to define lower and upper bounds for the auxiliary variables ξ_k and ζ_k , in order to completely define the expressions for the correspondent McCormick's envelopes. The first step for doing that is expressing ξ_k and ζ_k in polar coordinates:

$$\xi_k = V_k \cdot \cos\theta_k / V_k^2 = \cos\theta_k / V_k \quad (346)$$

$$\zeta_k = V_k \cdot \sin\theta_k / V_k^2 = \sin\theta_k / V_k \quad (347)$$

The determination of the maximum and minimum values of the above expressions is facilitated by the fact that the voltage magnitude V_k is strictly positive and bounded within $\underline{V}_k \leq V_k \leq \bar{V}_k$, and the voltage angles in typical distribution

systems assume values within a narrow interval around 0. Considering this, the following bounds may be defined:

$$\underline{\xi}_k = \cos[\max(|\underline{\theta}_k|, |\bar{\theta}_k|)]/\bar{V}_k \quad , \forall k \in \Omega_{GEN} \quad (348)$$

$$\bar{\xi}_k = 1/\underline{V}_k \quad , \forall k \in \Omega_{GEN} \quad (349)$$

$$\underline{\zeta}_k = \sin\underline{\theta}_k/\underline{V}_k \quad , \forall k \in \Omega_{GEN} \quad (350)$$

$$\bar{\zeta}_k = \sin\bar{\theta}_k/\underline{V}_k \quad , \forall k \in \Omega_{GEN} \quad (351)$$

4.3.3.2 Bounds for V_k and $I_{g,k}^{re}$ of a slack bus

In section 4.2.2.4.1, the product $V_k \cdot I_{g,k}^{re}$ has been approximated via a McCormick's envelope. The bounds for the voltage magnitude are usual inputs and need not to be discussed, and the procedure for estimating (conservative) bounds for $I_{g,k}^{re}$ has been discussed in section 4.3.2.3 – see equations (341) and (342).

4.4 An alternative MILP reformulation of the ACOPF in distribution systems

In the course of the research activities that led to this dissertation, an alternative MILP reformulation of the ACOPF in distribution systems has been investigated. This alternative formulation is similar to the one presented above in various aspects, but differs from it with respect to the construction of piecewise-linear approximations of non-linear, non-convex functions. In the formulation presented above, each segment of the piecewise-linear approximation of a non-linear function is a linear function, obtained by the affine combination of the vertices of the segment. In the alternative formulation, each segment of the piecewise-linear approximation represents a *constant value*, which is taken to be representative of the values that the non-linear function assumes between the vertices of a partition of its domain. This difference in the approximation of the non-linear functions requires the rewriting of several constraints of the ACOPF problem, particularly those that relate power injections at buses with the corresponding current injections.

This alternative formulation, which is thoroughly presented in Appendix B (chapter 8) for the sake of didactics, has been abandoned at early stages of the research activities due to its performance being inferior, with respect to accuracy and computational requirements, to the formulation presented in sections 4.1 to 4.3.

5 CASE STUDIES AND DISCUSSION OF RESULTS

In this chapter, the proposed MILP reformulation of the ACOPF is applied to a number of case studies. Two classes of case studies are considered:

- Those of section 5.1 allow the comparison of the solutions obtained with the proposed MILP reformulation of the ACOPF with the solutions obtained by exhaustive search, for the problem of network reconfiguration for the minimization of losses. While this comparison does not allow a thorough validation of the proposed formulation, due to the fact that only a parcel of its features is put into service, it serves the purpose of benchmarking its accuracy and computational performance.
- The case studies of section 5.2 illustrate the flexibility and the range of application of the MILP reformulation of the ACOPF. Each of the alternative objective functions (or modules for objective functions) presented in section 4.2.2 will be used in at least one application, with the exception of that presented in section 4.2.2.8.

For all applications of the proposed MILP reformulation of the ACOPF presented in this chapter, the methods described in section 4.3 have been used for obtaining the *disjunctive constants*, the *bounds for variables in McCormick's envelope*, and the sets of *evaluation points and evaluated values* for piecewise-linearizations with SOS2. The procedures described in subsection 4.3.2 have been employed considering $|\Gamma^{re}| = 5$ and $|\Gamma^{im}| = 9$ (subsection 4.3.2.1) and $|\Pi^{re}| = |\Pi^{im}| = 5$ (subsection 4.3.2.2). Furthermore, the range for the variation of the voltage angles of all buses in the system was assumed to be $-5^\circ \leq \theta_k \leq 5^\circ$ in all simulations, which is a conservative definition, as the numerical results will indicate.

5.1 Benchmark of the proposed formulation against an exhaustive search algorithm

In this section, the proposed MILP reformulation of the ACOPF is employed in four case studies, all of which involve the problem of network reconfiguration for the minimization of losses. The solutions obtained with the proposed MILP reformulation

of the ACOPF will be compared to those obtained by a brute force, exhaustive search method, with respect to accuracy and computational performance.

The two main reasons for choosing the problem of network reconfiguration for the analyses of this section are:

- If branch switching is the only control action modeled in a network reconfiguration problem, the problem will involve exclusively binary decisions. This facilitates the construction of an exhaustive search algorithm for the solution of the network reconfiguration problem, which will be necessary for the conducting the benchmarking activity that is the object of this section. Obviously, exhaustive search methods are excessively demanding when continuous decisions are taken into consideration.
- The network reconfiguration problem has been extensively dealt with in the technical literature. Thus, the input data associated with a number of test systems for network reconfiguration applications can be readily obtained, facilitating the construction of the case studies of this section.

The fact that network reconfiguration applications involve exclusively binary decisions prevents the full range of features of the proposed MILP formulation to be put into service. As mentioned in the introductory chapter of this dissertation, one of the main advantages of the proposed formulation is its flexibility to simultaneously represent discrete and continuous decisions. Thus, limiting the case studies of this section to network reconfiguration applications, while being necessary to allow the use of exhaustive search methods in manageable time, does not allow the validation of all features of the proposed formulation. Nonetheless, the comparison of the two solution methods will serve the purpose of providing insight on the accuracy and computational performance of the proposed MILP reformulation of the ACOPF, as well as on the adherence of the solutions to those obtained by exhaustive search, for a common problem in distribution system operations planning.

The four test systems considered in the benchmarking are presented in the following subsection. In section 5.1.2, the exhaustive search method employed for the benchmarking analysis is presented, and reference is made to the objective function used for the mixed-integer linear program. In section 5.1.3, the results of the simulations

conducted with the proposed MILP reformulation are compared to those obtained via exhaustive search.

For the network reconfiguration applications listed below, radiality of the network topology is required, and all buses are always required to be connected to the network. The objective of the reconfiguration is to minimize the total ohmic losses in the system. No costs are considered to be associated to switching actions.

5.1.1 Test systems

5.1.1.1 *Test system S1*

The data for test system S1 has been taken directly from [84].

This is the simplest test system, with 12 buses and 11 branches. All branches in the system may be switched. The input data for test system S1 is presented in Appendix A (section 7.1.1).

5.1.1.2 *Test system S2*

Test system S2 has also been taken from [84]. Slight modifications have been necessary to adjust the data to the format required by the ACOPF formulation: the addition of buses to allow the modeling of capacitor banks, and the addition of low-impedance branches ($R_{km} = 0$ p.u. and $X_{km} = 0.001$ p.u.) to connect these buses to the main system.

Test system S2 has 23 buses and 23 branches, 16 of which may be switched – the low-impedance branches used to connect the buses modeling capacitor banks are the only ones that cannot be switched. The input data for test system S2 is presented in Appendix A (section 7.1.2).

5.1.1.3 *Test system S3*

The data for test system S3 has been taken from [64].

Test system S3 has 33 buses and 37 circuits, all of which are considered as switchable in the original reference [64]. The exhaustive search of 2^{37} configurations could not be handled in manageable time (the brute force algorithm has been interrupted

after about 72 hours of computation). Therefore, in order to allow this test system to be treated by the brute force method, the number of switchable circuits has been reduced to 26 – i.e., 11 of the circuits closest to the *root node* have been considered as non-switchable.

The input data for this test system is presented in Appendix A (section 7.1.3).

5.1.1.4 *Test system S4*

The data for test system S4 was adapted from the *IEEE 123 Bus radial Distribution Feeder* presented in [85]. The original test system consisted of a three-phase unbalanced system, with structural and operational unbalance. Several modifications have been made in order to obtain a three-phase balanced distribution system with basis on the original data, as the proposed MILP reformulation of the ACOPF is currently limited to such systems. Despite the fact that the proposed MILP reformulation of the ACOPF explicitly models loads of the constant-current type (as case studies presented further in this document will show), these types of loads are not dealt with by the brute force algorithm employed for the benchmarking activity – thus, all loads of the constant-current type have been converted to constant-impedance loads. Voltage regulators were removed from the input data, and additional buses, connected to the system via low-impedance circuits, were added in order to model capacitor banks. Also, the total number of switchable branches in the system was increased from the original 11 to 16, in order to obtain a case with higher dimensions (notably, a higher number of feasible configurations to be investigated). With these modifications, the total number of buses and branches in the system is respectively 132 and 134.

A full description of the input data for test system S4 can be found in Appendix A (section 7.1.4).

5.1.2 Algorithm for exhaustive search and objective function for MILP approach

5.1.2.1 *Brute-force, exhaustive search algorithm*

The exhaustive search algorithm employed for the benchmarking analyses of this section is described in the following:

- (i) For a network with $|\Psi_{sw}|$ switchable circuits, there are $2^{|\Psi_{sw}|}$ possible network configurations to be investigated.
- (ii) Each of the $2^{|\Psi_{sw}|}$ configurations is first checked for connectivity and radiality. If the configuration is fully connected and radial (i.e., if there is a single path through which each bus is connected to one and only one root bus), the configuration is flagged as feasible with respect to connectivity and radiality.
- (iii) For all configurations that are feasible with respect to connectivity and radiality, the *backward-forward* load flow algorithm [67] is executed to solve for all complex bus voltages and branch currents in the system. The stop criterion for the execution of successive *backward-forward* iterations is that, from one iteration to another, the maximum variation in any component of any complex bus voltage does not exceed 10^{-5} p.u. Another stop criterion is that the number of iterations does not exceed 100 (though this did not happen in any of the simulations). After convergence, the losses in the system are calculated and stored.
- (iv) Once all $2^{|\Psi_{sw}|}$ configurations have been treated, all of the configurations for which the power flow problem has been solved are ordered, from that with the lowest losses to that with the highest losses. The one with the lowest losses is re-simulated, and the compliance of the solution to operational limits is checked. If the solution complies with operational limits, it is chosen as the optimal solution of the brute-force search. If not, the procedure is repeated with the next solution of the list, until a solution that complies with operational limits is found.

5.1.2.2 Objective function for MILP approach

In section 4.2.2.5, a number of alternative formulations for the objective function of the losses minimization problem have been presented. At this point, it is important to indicate which of these has been used for the benchmark of section 5.1, for the network reconfiguration problem.

As previously mentioned, the only control action considered for the case studies of this section is the switching of branches. As load shedding decisions are not

considered here, the products $V_k \cdot (1 - \rho_k)$ and $V_k^2 \cdot (1 - \rho_k)$ do not appear in the objective function. Furthermore, due to the fact that the brute force algorithm against which the MILP approach will be benchmarked does not treat loads of the *constant-current* type, the only loads that appear in the objective function are those of the constant-power and of the constant-impedance type.

It is necessary to model the dependence of the latter loads with the voltage magnitude of the buses to which they connect, and for that the approximation $V_k^2 \approx (2 \cdot V_k - 1)$ is employed. The reader will recall that this approximation is based on the truncated Taylor series calculated about the reference value $V_k^0 = 1$. Despite the fact that a piecewise-linear approximation of this product may be employed to ensure better control over the approximation accuracy, the results of section 5.1.3 will show that the truncated Taylor series technique is sufficient to ensure that the switching decisions taken with help of the MILP reformulation *perfectly* match those obtained with the brute force algorithm.

After this introduction, the exact objective function employed for the case studies of this section is presented:

$$z^{LOSS} = \min \left\{ \sum_{k \in \Omega_{ITFC}} V_k^{ref} \cdot I_{g,k}^{re} - \left[\sum_{k \in \Omega_{PCTE}} d_k^p + \sum_{k \in \Omega_{ZCTE}} (2 \cdot V_k - 1) \cdot \frac{R_k^l}{|Z_k^l|^2} \right] \right\} \quad (352)$$

The reader will notice that no specific costs have been assigned to losses, and therefore all results will be given in p.u. (for the tables of section 5.1.3, these will be converted to MW). The *connectivity approach* for the formulation of radiality constraints presented in section 4.2.1.7.1, approach (i), has been used for all case studies in this section.

5.1.3 Case study results

The main results of all case studies are summarized in Table 5.1.

Table 5.1. Case study results: benchmark of MILP formulation against exhaustive search

Test system			Execution time [s]			Switching decisions (status of switchable branches)		Comment
ID	Number of buses	Number of branches (switchable)	Brute force (BF)	MILP reform.	(MILP-BF)/BF [%]	Switched-on branches	Switched-off branches	
S1	12	11 (11)	1.14	1.69	48%	(1000-5), (5-4), (4-3), (3-2), (2-1), (1-101), (102-103), (103-104), (104-105),(105-2000)	(101-102)	Identical switching decisions in MILP and BF
S2	23	23 (16)	52.87	10.03	-81%	(1-4), (4-5), (4-6), (6-7), (2-8), (8-9), (9-12), (3-13), (13-14), (13-15), (15-16), (5-11), (10-14)	(8-10), (9-11), (7-16)	Identical switching decisions in MILP and BF
S3	33	37 (26)	65668	626.0	-99%	(5-6), (7-8), (9-10), (7-20), (10-11), (11-12), (12-13), (14-15), (15-16), (16-17) (20-21), (5-25), (25-26), (26-27), (27-28), (28-29) (29-30), (30-31), (8-14), (11-21), (17-32)	(6-7), (8-9), (13-14), (31-32), (24-28)	Identical switching decisions in MILP and BF
S4	132	134 (16)	651.1	688.9	5.8%	(42-44), (54-94), (25-44), (13-152), (60-160), (61-610), (97-197), (250-251), (450-451), (151-300), (300-350), (150-149), (33-149)	(23-25), (86-87) (18-135)	Identical switching decisions in MILP and BF

The computer used for all simulations is a Dell Vostro 3300 with the processor Intel® Core™ i5, with 2.26 GHz and 3.8 GB of usable RAM, and using Windows 7® as the operational system. The brute force algorithm has been coded and executed in Matlab® Version 7.10 64 bit. The MILP reformulation of the ACOPF has been coded and executed with FICO Xpress Mosel ® Version 3.2.2, with help of the graphical interface FICO Xpress-IVE ® Version 1.22.02, 64 bit.

As indicated in Table 5.1, identical switching decisions have been obtained with the MILP reformulation of the ACOPF and with the brute-force, exhaustive search algorithm, for all case studies. This effectively means that, if the optimal decisions obtained with both methods were implemented, the same ohmic losses in distribution network would be obtained – indicating that the *actual value* of the objective function obtained with the MILP formulation and the brute-force algorithm is identical, for all case studies.

Keeping in mind what has been discussed in the last paragraph, it is also worth comparing the *numerical value* of the objective function obtained by the MILP reformulation (i.e., the approximated numerical value corresponding to the solution of the mixed-integer program, and not the *actual value* that would be obtained by implementing the solution) to the numerical value of the objective function corresponding to the solution of the brute-force algorithm. This will provide the reader with insight on the accuracy of the approximations that are inherent to the MILP reformulation. The comparison of these values is shown in Table 5.2.

Table 5.2. Case study results: comparison of approximated numerical value corresponding to the solution of the mixed-integer program (MILP) to the numerical value of the objective function corresponding to the solution of the brute-force (BF)

Test system			Numerical value of total losses at optimal solution [MW]		
ID	Number of buses	Number of branches (switchable)	Brute force (BF)	MILP reform.	(MILP–BF)/BF [%]
S1	12	11 (11)	0.3297	0.3270	-0.8%
S2	23	23 (16)	0.4748	0.4814	1.4%
S3	33	37 (26)	0.1396	0.1458	4%
S4	132	134 (16)	0.0426	0.0452	6%

The results of Table 5.1 and Table 5.2 indicate that, even when the approximation of V_k^2 via the truncated Taylor series is used in the objective function, the results of the proposed MILP reformulation of the ACOPF closely match those obtained with the exhaustive search. The numerical value of the total losses obtained with the MILP reformulation differs slightly from that obtained with the brute force method, and the absolute value of this difference increases as the dimensions of the system increases. Systems S2 and S4 have loads of the constant-impedance type. Yet, the existence of loads of the constant-impedance type does not seem to be the most

preponderant factor for explaining the difference in the total losses obtained with the two methods – which is explained by the voltage magnitudes at all buses being close to 1.0 p.u. (the reference point for the approximation of V_k^2 via the truncated Taylor series).

For test system S1, the computational performance of the brute force algorithm is superior to that of the MILP reformulation, which is explained by the small dimensions of the system and by the fact that, due to the particular topology of this system (it basically corresponds to 12 buses sequentially and linearly aligned, as indicated by the data in Appendix A), very few of the topologies are feasible with respect to connectivity and have their power flow simulated. For test system S4, the computational performance of the exhaustive search method has been slightly superior to that of the MILP reformulation (5.8%). For the other systems, the computational performance of the MILP reformulation of the ACOPF has been superior to that of the brute force method. For test system S3, the system with the highest number of possible network configurations (2^{26}), the solution time with the MILP reformulation of the ACOPF was only 0.95% that of the brute force method. However, as the number of buses and branches increases and the number of possible configurations decreases from test system S3 to test system S4, the computational performance of the two methods become comparable – and, in fact, the brute force algorithm has a slightly better performance than the MILP reformulation. While analyzing this last result, the reader should keep in mind that, if continuous decisions were to be included in the case studies, the brute force algorithm could simply not be used, while the MILP reformulation would still apply.

For tests systems S1, S2 and S3, a further benchmark of the optimal solution obtained with the proposed MILP reformulation of the ACOPF for distribution systems can be made. The value of the ohmic losses obtained by evaluating the optimal solution obtained by the proposed formulation with a *backward-forward load flow*¹² (BFLF) is compared with the value of the optimal solution reported in the original references [84] (for test systems S1 and S2) and [64] (for test system S3). The original references have employed heuristics (variations of the *branch-exchange heuristic* presented in section

¹² This is the actual value of the system losses that would be obtained when the distribution system engineer implements the decision taken with support of the MILP formulation.

1.2, though this name is not actually used by the authors) for the solution of the network reconfiguration problem for the minimization of losses (except for the very simple system S1, for which an exhaustive search has been implicitly conducted in [84]). The results of the comparison are indicated in Table 5.3, from which is clear that the ohmic losses corresponding to the solution obtained with the proposed MILP formulation are inferior to those corresponding to the solution informed in the original references for test systems S2 (by 3.67%) and S3 (by 5.22%).

Table 5.3. Case study results: benchmark of optimal solution against solution informed in original references [84] (for test systems S1 and S2) and [64] (for test system S3)

Test system	Optimal solution obtained with MILP formulation		Optimal solution reported in original reference		(OPT-REF)/OPT [%]
	Switching decisions (status of switchable branches)	Total losses at optimal solution (evaluated with BFLF) [MW]	Switching decisions (status of switchable branches)	Total losses at optimal solution (evaluated with BFLF) [MW]	
	Switched-off branches		Switched-off branches		
S1	(101-102)	0.3297	(101-102)	0.3297	0%
S2	(8-10), (9-11), (7-16)	0.4748	(8-10), (5-11), (7-16)	0.4922	3.67%
S3	(6-7), (8-9), (13-14), (31-32), (24-28)	0.1396	(7,20), (8,14), (10,11), (27,28), (30,31)	0.1468	5.22%

Before moving on to the next section, it is worth providing the reader with insight on the actual intervals within which the voltage angles throughout the system have varied, for all simulated systems. The range of variation of voltage angles within the systems S1 to S4 is indicated in Table 5.4. From the table, it is clear that considering $-5^\circ \leq \theta_k \leq 5^\circ$ is a conservative modeling choice for all case studies.

Table 5.4. Case study results: range of variation of voltage angles across the systems S1 to S4.

Test system	Range of variation of voltage angles across the system [°]	
	Min	Max
S1	-1.24	0.00
S2	-1.65	0.00
S3	-1.02	0.60
S4	-2.11	0.00

5.2 Illustration of selected applications

In this section, five case studies, corresponding to different applications of the proposed MILP reformulation of the ACOPF, will be presented. Each of the alternative objective functions (or modules for objective functions) presented in section 4.2.2, with the exception of that presented in section 4.2.2.8, will be used in at least one application.

5.2.1 Application A1: emergency load shedding plan

For application A1, it is considered that the distribution operation planner wishes to construct an emergency load shedding plan for a severe contingency within the network of its utility. For that, it is necessary to determine which loads should be shed and which circuits should be maneuvered in the event of a specific, severe contingency, in order to minimize the combined costs of load shedding and ohmic losses.

The distribution system considered for this application, referred to as S5, was built upon the data previously defined for test system S3. However, in order to better reflect the actual conditions with which distribution system operators are faced, only a limited subset of branches is considered to be switchable. Furthermore, it is necessary to represent the severe contingency for which the load shedding plan is to be built. Thus, two branches relatively close to the step-down substation at the interface with the transmission system are removed from the data – the branches (5-6) and (5-25) from the original data for S3).

Also, the bus voltage magnitude limits, which in section 5.1.1.3 had been set to $0.90 \text{ p.u.} \leq V_k \leq 1.10 \text{ p.u.}$ to ensure the feasibility of the problem (the reader will recall that no load shedding actions were allowed for the analyses of section 5.1), are now set to $0.95 \text{ p.u.} \leq V_k \leq 1.05 \text{ p.u.}$, for all buses.

The distribution system operator is assumed to be able to shed 50% of the loads in the network – i.e., 16 loads have been randomly selected and marked as eligible for load shedding. The load shedding costs coefficients c_k^{SHED} vary within the interval [900 \$/MW, 1200 \$/MW], as shown in Appendix A. For the construction of the input data, the identification of the buses with loads that can be shed and the associated cost

coefficients were randomly sampled. The cost coefficient for ohmic losses equals 100 \$/MW.

The input data for system S5, used for application A1, are presented in detail in Appendix A (section 7.2.1). A schematic diagram of system S5 is shown in Figure 5.1. In this figure, the branches under contingency are not represented.

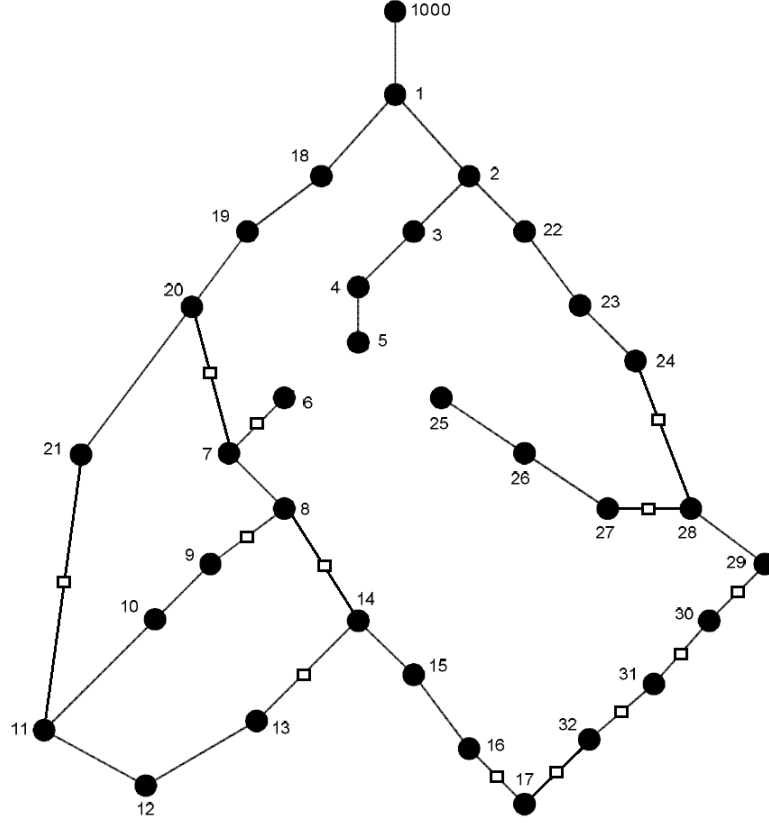


Figure 5.1: Schematic diagram of system S5. Switchable branches are indicated with a square. Adapted from [49].

The following objective function is employed for the minimization of the costs of load shedding and ohmic losses:

$$\begin{aligned}
 Z^{SHED \& \text{LOSSES}} = \min \{ & \sum_{k \in \Omega_{SHED}} c_k^{SHED} \cdot d_k^P \cdot \rho_k \\
 & + c_k^{LOSS} \cdot \{ \sum_{k \in \Omega_{ITFC}} V_k^{ref} \cdot I_{g,k}^{re} \\
 & - [\sum_{k \in \{ \Omega_{PCTE} \cap \Omega_{NSHD} \}} d_k^P + \sum_{k \in \{ \Omega_{PCTE} \cap \Omega_{SHED} \}} d_k^P \cdot (1 - \rho_k)] \} \} \quad (353)
 \end{aligned}$$

In the following three subsections, we present the results of the application of the proposed MILP reformulation under consideration of each of the three connectivity

approaches presented in subsection 4.2.1.7. As mentioned in section 4.2.1.7, the three approaches differ among themselves with respect to the necessity (or possibility) of entirely removing from the network a bus whose load has been shed.

5.2.1.1 Simulation considering connectivity approach (i)

In this subsection, the connectivity approach presented in section 4.2.1.7.1 is considered while determining the emergency load shedding plan for system S5. The reader will recall that, in this approach, is considered that all buses of the distribution system must be connected to the network at all times.

The optimal emergency load shedding plan, obtained by the solution of the corresponding mixed-integer linear program, is summarized in Table 5.5. With help of Figure 5.1, it is easily understood that, after the maneuvering decisions are taken into account, the buses with the lowest voltage magnitudes are located at the extremities of the feeder. Among the curtailable loads located near these buses, those with the lowest value of c_k^{SHED} are shed in order to achieve compliance to the admissible range of bus voltage magnitudes.

Table 5.5. Case study results: application A1, emergency load shedding plan, approach (i).

Test system	Objective function			Shed loads			Switching decisions (status of switchable branches)		Information on bus voltage magnitude		Execution time [s]
	Objective function [\$]	Load shedding costs [\$]	Costs of ohmic losses [\$]	Active power [MW]	Connected to bus	Cost coefficient of load shedding [\$/MW]	Switched-on branches	Switched-off branches	Buses with lowest voltage magnitude in system (magnitude [p.u.])	Buses with highest voltage magnitude in system (magnitude [p.u.])	
S5	248.0	237.0	11.0	0.06	15	950	(6, 7), (7, 20), (8, 14), (11, 21), (16, 17), (27, 28), (29, 30), (30, 31), (31, 32), (24, 28)	(8-9), (13-14), (17-32)	17 (0.951), 32 (0.951)	1000 (1.00), 2 (0.997)	242.9
				0.2	29	900					

The operation point corresponding to the optimal solution indicated in Table 5.5 has been used as the input data for a *backward-forward* load flow [67] simulation, for system S5. The stop criterion for the execution of successive *backward-forward* iterations is that the maximum variation in any component of any complex bus voltage does not exceed 10^{-5} p.u. from one iteration to another. The losses obtained by the backward-forward load flow simulation are 0.22% lower than those obtained with the MILP reformulation of the ACOPF. Furthermore, it is relevant to quantify the approximation errors of the bus voltages. Table 5.6 also indicates the results obtained by the *backward-forward* load flow, as well as the relative error between the voltages obtained by the MILP problem and the backward-forward load flow. It is clear that the approximation errors range from 0.00005% to 0.0006% for voltage magnitudes and from -1.13% to -0.04% for bus angles (excluding the reference voltage bus). At this point, the reader is reminded that, as the branch impedance is known for every branch in the system, the branch currents can be readily calculated when the information of the bus voltage magnitudes is at hand – i.e., the complex bus voltages are the *state*

variables of the system. Due to that, an option is made not to construct a table similar to Table 5.6 for branch currents.

Table 5.6. Bus voltages: solution of MILP ACOPF (MILP), simulation with backward-forward load flow (BFLF) and comparison of relative error, given by (MILP-BFLF)/BFLF.

Bus #	Voltage magnitude [p.u.]			Voltage angle [°]		
	BFLF	MILP	(MILP-BFLF)/BFLF [%]	BFLF	MILP	(MILP-BFLF)/BFLF [%]
1000	1.00000	1.00000	-	0.00000	0.00000	-
1	0.99743	0.99743	0.00005%	-0.00168	-0.00166	-1.1270%
2	0.98955	0.98955	0.0002%	-0.00648	-0.00642	-0.8441%
3	0.98885	0.98884	0.0002%	-0.00544	-0.00538	-1.0227%
4	0.98850	0.98849	0.0002%	-0.00698	-0.00693	-0.7902%
5	0.98810	0.98809	0.0003%	-0.01652	-0.01646	-0.3106%
6	0.95875	0.95875	0.0001%	-0.75173	-0.75121	-0.0692%
7	0.95983	0.95983	0.0001%	-0.76110	-0.76060	-0.0665%
8	0.95757	0.95757	0.0000%	-0.80262	-0.80209	-0.0658%
9	0.96144	0.96144	0.0003%	-0.66929	-0.66881	-0.0713%
10	0.96153	0.96152	0.0003%	-0.66930	-0.66882	-0.0713%
11	0.96182	0.96182	0.0004%	-0.67151	-0.67104	-0.0703%
12	0.95924	0.95923	0.0003%	-0.68663	-0.68613	-0.0732%
13	0.95844	0.95844	0.0004%	-0.70305	-0.70253	-0.0728%
14	0.95390	0.95390	0.0001%	-0.91218	-0.91161	-0.0629%
15	0.95295	0.95295	0.0001%	-0.92668	-0.92610	-0.0625%
16	0.95100	0.95100	0.0002%	-0.99801	-0.99740	-0.0611%
17	0.95042	0.95042	0.0002%	-1.00687	-1.00625	-0.0611%
18	0.99530	0.99530	0.0001%	-0.04075	-0.04071	-0.1076%
19	0.97734	0.97734	0.0002%	-0.34219	-0.34192	-0.0797%
20	0.97237	0.97237	0.0002%	-0.46978	-0.46944	-0.0722%
21	0.96891	0.96891	0.0003%	-0.56019	-0.55981	-0.0680%
22	0.98312	0.98311	0.0003%	-0.06009	-0.06000	-0.1353%
23	0.97055	0.97054	0.0005%	-0.22200	-0.22187	-0.0581%
24	0.96136	0.96136	0.0005%	-0.33918	-0.33901	-0.0493%
25	0.95532	0.95532	0.0005%	-0.46205	-0.46185	-0.0428%
26	0.95546	0.95545	0.0005%	-0.46143	-0.46123	-0.0428%
27	0.95659	0.95659	0.0005%	-0.43832	-0.43813	-0.0436%
28	0.95786	0.95785	0.0006%	-0.41108	-0.41089	-0.0450%
29	0.95611	0.95610	0.0005%	-0.41169	-0.41150	-0.0451%
30	0.95209	0.95209	0.0004%	-0.49015	-0.48994	-0.0422%
31	0.95121	0.95121	0.0004%	-0.51156	-0.51135	-0.0415%
32	0.95094	0.95094	0.0003%	-0.51874	-0.51853	-0.0413%

5.2.1.2 Simulation considering connectivity approach (ii)

In connectivity approach (ii), which has been presented in section 4.2.1.7.2, it is considered that the load and/or generator at a bus can only be de-energized if all circuits that connect to that bus are removed from the network. Removing a bus from the network requires that all circuits connected to that bus are deactivated (switched-off).

The optimal emergency load shedding plan obtained by the solution of the corresponding mixed-integer linear program which has been formulated considering the second approach to connectivity requirements is summarized in Table 5.7.

Table 5.7. Case study results: application A1, emergency load shedding plan, approach (ii).

Test system	Objective function			Shed loads			Switching decisions (status of switchable branches)		Information on bus voltage magnitude		Execution time [s]
	Objective function [\$]	Load shedding costs [\$]	Costs of ohmic losses [\$]	Active power [MW]	Connected to bus	Cost coefficient of load shedding [\$/MW]	Switched-on branches	Switched-off branches	Buses with lowest voltage magnitude (magnitude [p.u.])	Buses with maximum voltage magnitude (magnitude [p.u.])	
S5	543.3	532.5	10.8	0.09	17	1150	(6-7), (7-20), (8-14), (27-28), (11-21), (24-28)	(8-9), (13-14), (16-17), (29-30), (30-31), (31-32), (17-32)	(25) 0.954, (26) 0.954	1000 (1.00), 2 (0.997)	11.03
				0.15	30	950					
				0.21	31	1050					
				0.06	32	1100					

From Table 5.7, it is clear that considering that a load can only be shed if its bus is removed from the network leads to an emergency load shedding plan with higher costs than that of subsection 5.2.1.1. The reader will notice that it is now not possible to shed the loads at buses 15 and 29 (shedding these two buses corresponds to the optimal solution obtained in subsection 5.2.1.1).

Analogously to what has been done in subsection 5.2.1.1, the operation point corresponding to the optimal solution indicated in Table 5.7 has been used as the input data for a *backward-forward* load flow [67] simulation. The losses obtained by the

backward-forward load flow simulation are 0.17% lower than those obtained with the MILP reformulation of the ACOPF. Analogously to what was done in section subsection 5.2.1.1, it is relevant to quantify the approximation errors of bus voltages. The results of this comparison are shown in Table 5.8. It is clear that the approximation errors range from 0.00003% to 0.0008% for voltage magnitudes (excluding the reference voltage bus) and from -0.24% to +0.31% for bus angles.

Table 5.8. Bus voltages: solution of MILP ACOPF (MILP), simulation with backward-forward load flow (BFLF) and comparison of relative error, given by (MILP-BFLF)/BFLF.

Bus #	Voltage magnitude [p.u.]			Voltage angle [°]		
	BFLF	MILP	(MILP-BFLF)/BFLF [%]	BFLF	MILP	(MILP-BFLF)/BFLF [%]
1000	1.00000	1.00000	-	0.00000	0.00000	-
1	0.99747	0.99747	0.00003%	0.01471	0.01474	0.2250%
2	0.98965	0.98965	0.0002%	0.09952	0.09966	0.1392%
3	0.98894	0.98894	0.0003%	0.10056	0.10070	0.1403%
4	0.98859	0.98859	0.0003%	0.09901	0.09915	0.1422%
5	0.98819	0.98819	0.0002%	0.08948	0.08962	0.1540%
6	0.96019	0.96018	0.0005%	-0.72006	-0.71951	-0.0756%
7	0.96126	0.96126	0.0006%	-0.72940	-0.72887	-0.0724%
8	0.95931	0.95931	0.0005%	-0.77023	-0.76968	-0.0719%
9	0.96220	0.96219	0.0006%	-0.64557	-0.64508	-0.0767%
10	0.96228	0.96228	0.0006%	-0.64559	-0.64509	-0.0764%
11	0.96258	0.96257	0.0006%	-0.64779	-0.64731	-0.0754%
12	0.95999	0.95999	0.0007%	-0.66289	-0.66237	-0.0786%
13	0.95920	0.95920	0.0006%	-0.67928	-0.67875	-0.0781%
14	0.95631	0.95630	0.0005%	-0.87156	-0.87096	-0.0687%
15	0.95558	0.95558	0.0005%	-0.88546	-0.88485	-0.0687%
16	0.95485	0.95485	0.0004%	-0.91581	-0.91519	-0.0677%
18	0.99540	0.99540	0.0001%	-0.02385	-0.02379	-0.2437%
19	0.97793	0.97793	0.0003%	-0.32131	-0.32102	-0.0909%
20	0.97312	0.97311	0.0005%	-0.44637	-0.44602	-0.0798%
21	0.96966	0.96965	0.0006%	-0.53665	-0.53625	-0.0741%
22	0.98308	0.98307	0.0004%	0.13544	0.13567	0.1708%
23	0.97000	0.96999	0.0007%	0.16305	0.16346	0.2565%
24	0.96032	0.96031	0.0007%	0.23882	0.23942	0.2506%
25	0.95372	0.95371	0.0007%	0.23425	0.23498	0.3091%
26	0.95386	0.95385	0.0008%	0.23487	0.23559	0.3076%
27	0.95499	0.95499	0.0007%	0.25806	0.25877	0.2725%
28	0.95626	0.95625	0.0007%	0.28539	0.28607	0.2383%
29	0.95458	0.95458	0.0006%	0.38439	0.38511	0.1858%

5.2.1.3 *Simulation considering connectivity approach (iii)*

The third approach to connectivity requirements has been presented in section 4.2.1.7.3. In this approach, it is considered that the buses to which loads that are shed and generators that are curtailed, as well as all buses that do have any fixed or curtailable injections, may or may not be disconnected from the network, according to the impacts of their connection or disconnection on the objective function.

For system S5, approach (iii) leads to exactly the same results as approach (i). It is worth mentioning that, despite of the third approach having led to the same results as the first for this particular system, the results obtained with both approaches may differ for other applications and systems.

5.2.2 Application A2: generation curtailment at light loading hours

For this second application, the distribution system expansion planner is assumed to have a list of requests for the connection of renewable generators to the distribution system, and must determine which of those requests to accept and which to decline, so that the maximum amount of renewable generation can be connected to the system while ensuring adequate technical conditions – i.e., compliance with bus voltage and branch current magnitude limits. This problem is equivalent to minimizing the curtailment of generators with non-controllable active power output (the *curtailed* generators are those whose connection request will not be met), with the curtailment cost coefficient set to unity for all generators.

As well as in the previous section, the data for the distribution system considered for this application, referred to as S6, have also been obtained by modification of the input data for test system S3. It assumed here that the most critical condition for the evaluation of the connection of the renewable generators is at night, when the voltage magnitude at the interface with the transmission system is high, the load within the distribution network is low, and the generation is high due to the dynamics of the non-controllable primary energy resources. The modifications made for obtaining system S6 will reflect this assumption. These modifications are listed in the following:

- (i) Only 11 of the 37 branches in the system are considered to be switchable;
- (ii) The load at all buses is reduced to 30% of their original value;
- (iii) The voltage magnitude of the bus at the interface with the transmission system is considered to be of 1.05 p. u.;
- (iv) The admissible range for the voltage magnitude of all buses in the system is set to $0.95 \text{ p. u.} \leq V_k \leq 1.05 \text{ p. u.}$;
- (v) Candidate renewable generators are assigned to 20 of the 33 buses in the system.
- (vi) The active power output of each of these generators has been randomly sampled from the interval [50 kW, 150 kW] (considering an uniform probability distribution).
- (vii) It is assumed that each generator can control its power factor within the range $[0.98_{\text{lagging}}, 0.98_{\text{leading}}]$ (considering the installed active power capacity). In reality, this is a very narrow power factor for many common distributed generation technologies. Nonetheless, this narrow power factor is assumed to make the analysis scenario somewhat more complex.

In this application, the “*generation curtailment*” decision does not refer to the actual physical disconnection of the generation from the system, but rather to the denial of a connection request. Due to that, the first approach to connectivity requirements, which has been presented in subsection 4.2.1.7.1, will be considered. In fact, this first approach will be considered for all applications presented in the following subsections, except for that of subsection 5.2.4.

The input data for system S6, used for application A2, are presented in detail in Appendix A (section 7.2.2). A schematic diagram of system S5 is shown in Figure 5.2.

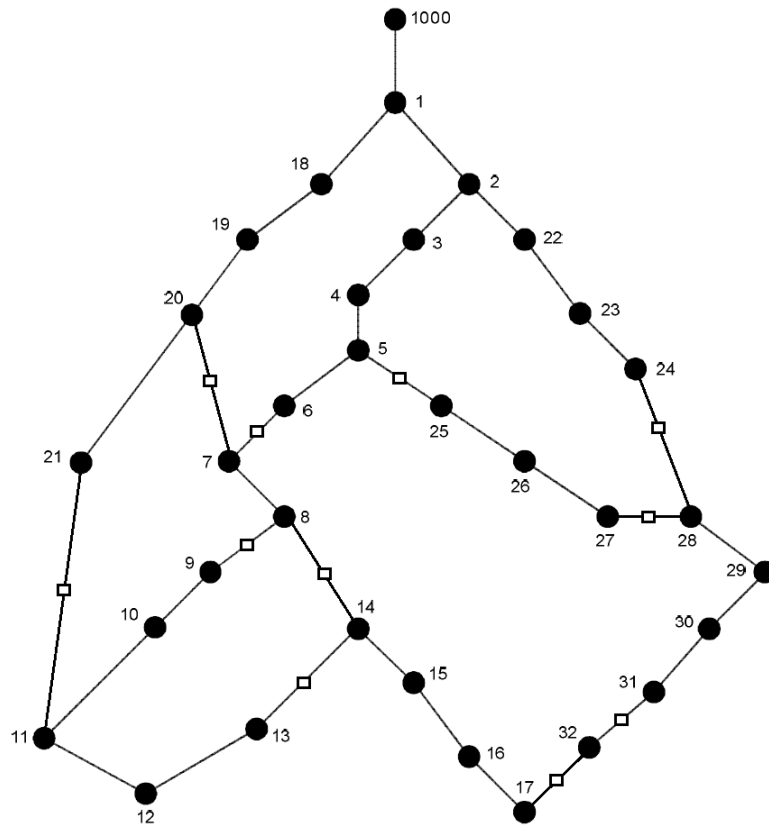


Figure 5.2: Schematic diagram of system S6. Switchable branches are indicated with a square.
Adapted from [49].

The objective function used in this application corresponds exactly to equation (197) of section 4.2.2.

The set of answers to connection requests that result in the highest amount of renewable generation connected to system S6 are indicated in Table 5.9.

Table 5.9. Case study results: application A2, generation curtailment at light loading hours.

Test system	Objective function	Answer to connection requests from renewable generators				Switching decisions (status of switchable branches)		Execution time [s]
	Minimum amount of active power generation curtailed [kW]	Connection authorized		Connection not authorized		Switched-on branches	Switched-off branches	
		Active power [kW]	Connected to bus	Active power [kW]	Requested connection to bus			
S6	326	90	2	58	6	(8-9), (27-28), (31-32), (8-14), (11-21), (17-32)	(6-7), (13-14), (5-25), (7-20), (24-28)	1602.84
		89	3	100	18			
		62	7	168	21			
		138	8					
		68	9					
		88	11					
		65	12					
		75	15					
		69	19					
		146	22					
		123	23					
		153	25					
		126	28					
		85	29					
		94	30					
89	31							
114	32							

Due to the high voltage at the bus at the interface with the transmission system, the light loading conditions and the fact that the generators may only vary their power factor within a very limited range, the voltage profile within the network of system S6 is very high. In fact, it is the need to prevent voltages above 1.05 p.u. that leads to the curtailment (i.e., denial of the connection request) of the three generators indicated in Table 5.9.

The bus voltages corresponding to the optimal solution of the MILP reformulation of the ACOPF are indicated in Table 5.10. In order to quantify the approximation errors of the bus voltage magnitudes, the operating point corresponding to the optimal solution of the MILP problem has been used as input data for a simulation using the *backward-forward* load flow algorithm [67] (i.e., the network topology and all bus injections, including the active power absorbed by generators, which have been modeled as fixed values, were used as input data and the system has

been simply simulated). The stop criterion for the execution of successive *backward-forward* iterations is that the maximum variation in any component of any complex bus voltage does not exceed 10^{-5} p.u. from one iteration to another. Table 5.10 also indicates the results obtained by the *backward-forward* load flow, as well as the relative error between the voltages obtained by the MILP problem and the backward-forward load flow. The approximation errors range from 0.0002% to 0.0022% (excluding the reference voltage bus, for which the voltage magnitude is fixed in application A2) for voltage magnitudes, and from $2 \cdot 10^{-10}$ (for bus 11) to 0.02% for bus angles (also excluding the reference voltage bus).

Table 5.10: Bus voltage magnitude profile: solution of MILP ACOPT (MILP), simulation with backward-forward load flow (BFLF) and comparison of relative error, given by (MILP-BFLF)/BFLF.

Bus #	Voltage magnitude [p.u.]			Voltage angle [°]		
	BFLF	MILP	(MILP-BFLF)/BFLF [%]	BFLF	MILP	(MILP-BFLF)/BFLF [%]
1000	1.05000	1.05000	-	0.00000	0.00000	-
1	1.04999	1.04999	0.00038%	0.03986	0.03986	0.0016%
2	1.04955	1.04955	0.0005%	0.08979	0.08978	0.0069%
3	1.04936	1.04936	0.0004%	0.09752	0.09752	0.0072%
4	1.04909	1.04909	0.0005%	0.09704	0.09704	0.0071%
5	1.04856	1.04856	0.0003%	0.08871	0.08870	0.0077%
6	1.04839	1.04839	0.0004%	0.07846	0.07845	0.0091%
7	1.04775	1.04774	0.0008%	2.50050	2.50053	0.0014%
8	1.04793	1.04791	0.0015%	2.48559	2.48563	0.0015%
9	1.04733	1.04731	0.0014%	2.18062	2.18063	0.0006%
10	1.04692	1.04691	0.0009%	2.13257	2.13257	0.0002%
11	1.04619	1.04618	0.0006%	2.04040	2.04040	0.0000%
12	1.04596	1.04595	0.0005%	2.06733	2.06733	0.0001%
13	1.04574	1.04573	0.0007%	2.06319	2.06320	0.0004%
14	1.04671	1.04669	0.0018%	3.03329	3.03336	0.0022%
15	1.04693	1.04692	0.0010%	3.21596	3.21604	0.0025%
16	1.04496	1.04495	0.0011%	3.59062	3.59072	0.0028%
17	1.04502	1.04501	0.0012%	3.76136	3.76148	0.0032%
18	1.04982	1.04982	0.0002%	0.10678	0.10678	0.0003%
19	1.04908	1.04907	0.0007%	0.71314	0.71313	0.0002%
20	1.04840	1.04839	0.0006%	0.88890	0.88889	0.0003%
21	1.04697	1.04697	0.0003%	1.21826	1.21825	0.0009%
22	1.04918	1.04918	0.0002%	0.11662	0.11661	0.0107%
23	1.04788	1.04787	0.0005%	0.12922	0.12920	0.0152%
24	1.04695	1.04695	0.0002%	0.11795	0.11793	0.0171%
25	1.04492	1.04490	0.0016%	4.34433	4.34450	0.0038%
26	1.04472	1.04470	0.0021%	4.33435	4.33452	0.0038%
27	1.04424	1.04422	0.0018%	4.28260	4.28276	0.0038%
28	1.04398	1.04396	0.0021%	4.24614	4.24631	0.0039%

Bus #	Voltage magnitude [p.u.]			Voltage angle [°]		
	BFLF	MILP	(MILP-BFLF) /BFLF [%]	BFLF	MILP	(MILP-BFLF) /BFLF [%]
29	1.04356	1.04354	0.0022%	4.21362	4.21379	0.0041%
30	1.04403	1.04401	0.0017%	4.05139	4.05153	0.0033%
31	1.04427	1.04425	0.0020%	3.98601	3.98614	0.0033%
32	1.04491	1.04489	0.0016%	3.89300	3.89312	0.0031%

5.2.3 Application A3: minimization of the sum of variable generation costs and costs of power imports

For this third application, it is assumed that the distribution system operator wishes to minimize the total costs related to supplying active power to the distribution system, including the variable costs of generators whose active power output can be controlled and the costs of power imports.

The distribution system considered for this third application, referred to as S7, was also obtained by modifying the input data for test system S3. The following modifications have been made to obtain S7:

- (i) Only 11 of the 37 branches in the system are considered to be switchable;
- (ii) The load at all buses are increased to 110% of their original value;
- (iii) The voltage at the interface with the transmission system is assumed to be of 1.025 p.u. and the admissible range for the voltage magnitude of all buses in the system is set to $0.95 \text{ p.u.} \leq V_k \leq 1.05 \text{ p.u.}$;
- (iv) It is assumed that there are four generations with controllable active power output in the system. Those generators are assumed to be connected to buses 7, 14, 17 and 29. The installed capacity of the generator connected to bus 7 is of 500 kW, and all other generators have an installed capacity of 300 kW. Each generator able to control its power factor within the range $[0.95_{\text{lagging}}, 1]$ (considering the installed active power capacity).
- (v) The unitary production costs for the generators connected to buses 7, 14, 17 and 29 are respectively of 110 \$/MWh, 118 \$/MWh,

118 \$/MWh and 145 \$/MWh. The costs of imports from the external network are of 115 \$/MWh.

- (vi) A period of one hour is considered, so that the conversion from MWh to MW is immediate.

The input data for system S7, used for application A3, are presented in detail in Appendix A (section 7.2.3). The network topology of S7 is identical to that indicated in Figure 5.2.

The following objective function is employed for the minimization of the sum of variable generation costs and costs of power imports:

$$Z^{GEN \&IMPORT} = \min \left\{ \sum_{k \in \Omega_{CTRPQ}} C_k^{GEN} \cdot g_k^P + \sum_{k \in \Omega_{ITFC}} C_k^{IMPORT} \cdot V_k^{ref} \cdot I_{g,k}^{re} \right\} \quad (354)$$

Two distinct groups of simulations will be executed for system S7. For the first group of simulations, whose results are indicated in subsection 5.2.3.1, it is required that the system is operated in a radial fashion (i.e., the radiality constraints are enforced). For the second group of simulations, whose results are indicated in subsection 5.2.3.2, the system may be operated in a meshed fashion if this is the optimal configuration (i.e., the radiality constraints are not enforced).

5.2.3.1 Radiality enforced

For the simulations of this section, it has been considered that the distribution system S7 must be operated in a radial fashion – i.e., radiality constraints are enforced while solving the problem of minimization of supply costs.

For the first simulation, the formulation of the constraints for obtaining the generator currents that was presented in section 4.2.1.2 has been used. The reader will recall that this formulation makes use of McCormick's envelopes for modeling bilinear products – and for that reason we identify this formulation as *formulation with McCormick's envelopes*. The corresponding results are shown in Table 5.11.

Table 5.11. Case study results: application A3 with radiality enforced, minimization of the sum of variable generation costs and costs of power imports, *formulation with McCormick's envelopes*.

Test system	Objective function			Variable generation within distribution network			Switching decisions (status of switchable branches)		Execution time [s]
	Objective function [\$]	Variable generation costs [\$]	Power import costs [\$]	Active power [kW]	Unitary production costs [\$/MW]	Connected to bus	Switched-on branches	Switched-off branches	
S7	476.5	103.9	372.6	0.5	110	7	(5-25), (8-14), (27-28), (31-32), (11-21), (17-32)	(6-7), (8-9), (7-20), (13-14), (24-28)	45.3
				0.200	118	14			
				0.215	118	17			
				0	145	29			

It is clear that the location of the generators within the network influences the dispatch decisions, mainly due to the avoidance of ohmic losses – this being the main reason for the generators at buses 14 and 17 having a non-zero dispatch, despite the fact that their unitary production costs (118 \$/MW) is superior to the unitary costs of power imports (115 \$/MW).

With the results of the first simulation at hand, a second simulation has been conducted. For this second simulation, the formulation of the constraints for obtaining the generator currents presented in Appendix C (section 9.2) has been used. The reader will recall that this formulation completely eliminates the need to employ McCormick's envelopes, as the generator currents $I_{g,k}^{re}$ and $I_{g,k}^{im}$ are treated as non-linear functions of four decision variables – i.e., $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$ and $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$ – and piecewise-linear approximations of these functions are constructed with help of SOS2. For the construction of these piecewise-linear functions, the cardinality of the sets T^Q and T^P (see section 9.2) has been defined as $|T^Q| = |T^P| = 5$, with the evaluation points distributed equally within the allowable range for the active and reactive power outputs. The formulation used in the second simulation is identified as *formulation with piecewise-linear approximations* in the tables of this section. The results obtained when using this formulation are shown in Table 5.12.

Table 5.12. Case study results: application A3 with radiality enforced, minimization of the sum of variable generation costs and costs of power imports, *formulation with piecewise-linear approximations*.

Test system	Objective function			Variable generation within distribution network			Switching decisions (status of switchable branches)		Execution time [s]
	Objective function [\$]	Variable generation costs [\$]	Power import costs [\$]	Active power [kW]	Unitary production costs [\$/MW]	Connected to bus	Switched-on branches	Switched-off branches	
S7	479.1	118.3	360.8	0.5	110	7	(5-25), (8-14), (27-28), (31-32), (11-21), (17-32)	(6-7), (8-9), (7-20), (13-14), (24-28)	811
				0.241	118	14			
				0.296	118	17			
				0	145	29			

It is clear that the network topology (switching decisions) obtained with both formulations is equal.

By comparing the results of Table 5.11 and Table 5.12, it may seem at first that the solution obtained with the *formulation with McCormick's envelopes* (Table 5.11) is better than that obtained with the *formulation with piecewise-linear approximations* (Table 5.12), due to the numerical value of the objective function of the former being inferior to that of the latter. However, the approximations of generator currents in the first formulation are presumably less accurate than that of the second formulation, which may indicate that, if the distribution system operator were to implement the generation dispatch of both formulations, the actual power supply costs associated with the first solution ($g_k^7 = 0.5$ MW, $g_k^{14} = 0.2$ MW, $g_k^{17} = 0.215$ MW, $g_k^{29} = 0$ MW, with the remainder of the power requirements supplied by power imports) could be higher than those of the second solution ($g_k^7 = 0.5$ MW, $g_k^{14} = 0.241$ MW, $g_k^{17} = 0.296$ MW, $g_k^{29} = 0$ MW, with the remainder of the power requirements supplied by power imports).

In order to investigate the hypothesis presented in the last paragraph, both solutions have been used as inputs for a *backward-forward power flow simulation* [67]: the decisions regarding the network topology (switching decisions) and generation dispatch (active power output of generators connected to buses 7, 14, 17 and 29) have been used as fixed input data, and the amount of power imports corresponding to each solution (*formulation with McCormick's envelopes* and *formulation with piecewise-*

linear approximations) have been obtained by the solution of the *backward-forward power flow*. With these results at hand, the value of the *actual system operation costs* can be calculated. Furthermore, the value of the actual system operating costs obtained by the procedure described above have been compared to that corresponding to the solution of a non-linear AC optimal power flow problem (NL-ACOPF), obtained with help of the software OptFlow [87]. This NL-ACOPF software does not support discrete decisions, and therefore the network topology corresponding to the switching decisions specified in the previous tables has been considered as fixed.

The system operating costs obtained with help of the procedure described in the two previous paragraph, for the three situations (simulations of the *actual system operation costs* associated with the solution obtained by the MILP *formulation with McCormick's envelopes* and with the solution obtained by the MILP *formulation with piecewise-linear approximations*, as well as the system operation costs obtained with the NL-ACOPF), are indicated in Table 5.13.

Table 5.13. Comparison of solutions for application A3, with radiality enforced: simulation of *actual system operation costs* associated with the solutions obtained by the MILP *formulation with McCormick's envelopes* and by the MILP *formulation with piecewise-linear approximations*, as well as the operating costs obtained with the NL-ACOPF with the network topology considered as fixed.

Item		Simulation of solutions obtained by the MILP formulation, with backward-forward load flow		Solution with NL-ACOPF (benchmark)
		Formulation with McCormick's envelopes	Formulation with piecewise-linear approximations	
Active power output [MW]	Generator at bus 7	0.500	0.500	0.500
	Generator at bus 14	0.212	0.241	0.288
	Generator at bus 17	0.226	0.296	0.299
	Generator at bus 29	0	0	0
Power imports [MW]		3.29	3.15	3.09
Actual system operation costs [\$]		485.2	480.3	479.3

The results of Table 5.13 indicate that the actual system operation costs obtained with the *formulation with piecewise-linear approximations* are inferior to those obtained with the *formulation with McCormick's envelopes*. In fact, the system operation costs obtained with the *formulation with piecewise-linear approximations* are only 0.21% higher than those associated with the benchmark solution (that obtained with the NL-ACOPF, considering the network topology as fixed). It is also evident that the

generation dispatch decisions obtained with the *formulation with piecewise-linear approximations* are closer to these obtained with the benchmark solution. The reader should notice, however, that the execution time for the *formulation with piecewise-linear approximations* (811 s) is considerably higher than the execution time for the *formulation with McCormick's envelopes* (45.3 s). Thus, the *trade-off* between approximation accuracy and computational performance becomes evident.

5.2.3.2 Meshed operation allowed

For the simulations of this section, it has been considered that the distribution system S7 may be operated either radially or in a meshed fashion – i.e., radiality constraints are *not* enforced while solving the problem.

For the first simulation, the formulation of the constraints for obtaining the generator currents that was presented in section 4.2.1.2 has been used (*formulation with McCormick's envelopes*). The corresponding results are shown in Table 5.17.

Table 5.14. Case study results: application A3 with meshed operation allowed, minimization of the sum of variable generation costs and costs of power imports, *formulation with McCormick's envelopes*.

Test system	Objective function			Variable generation within distribution network			Switching decisions (status of switchable branches)		Execution time [s]
	Objective function [\$]	Variable generation costs [\$]	Power import costs [\$]	Active power [kW]	Unitary production costs [\$/MW]	Connected to bus	Switched-on branches	Switched-off branches	
S7	474.5	95.1	379.4	0.5	110	7	(6-7),(5-25), (8-14), (7-20),(27-28), (31-32),(11-21), (17-32),(24-28)	(8-9), (13-14)	152.0
				0.161	118	14			
				0.179	118	17			
				0	145	29			

The first noticeable result is that, now that the radiality constraints have been relaxed, the optimal network configuration corresponds to a meshed one.

With the results of the first simulation at hand, a second simulation has been conducted. For this second simulation, the formulation of the constraints for obtaining the generator currents presented in Appendix C (section 9.2) has been used, with

$|T^Q| = |T^P| = 5$. This second formulation is identified as *formulation with piecewise-linear approximations* in the following tables. The results obtained when using this formulation are shown in Table 5.15.

Table 5.15. Case study results: application A3 with meshed operation allowed, minimization of the sum of variable generation costs and costs of power imports, *formulation with piecewise-linear approximations*.

Test system	Objective function			Variable generation within distribution network			Switching decisions (status of switchable branches)		Execution time [s]
	Objective function [\$]	Variable generation costs [\$]	Power import costs [\$]	Active power [kW]	Unitary production costs [\$/MW]	Connected to bus	Switched-on branches	Switched-off branches	
S7	478.1	125.8	352.3	0.5	110	7	(6-7),(5-25), (8-14), (7-20),(13-14), (27-28), (31-32),(11-21), (17-32),(24-28)	(8-9)	1414.3
				0.300	118	14			
				0.300	118	17			
				0	145	29			

The reader will notice that the network topology of the solution obtained with the *formulation with piecewise-linear approximations* (Table 5.15) differs from that corresponding to the solution obtained with the *formulation with McCormick's envelopes* (Table 5.17). The network topology indicated in Table 5.15 is also a meshed one.

Again, by comparing the results of Table 5.17 and Table 5.15, it may seem at first that the solution obtained with the *formulation with McCormick's envelopes* (Table 5.17) is better than that obtained with the *formulation with piecewise-linear approximations* (Table 5.15), due to the value of the objective function of the former being inferior to that of the latter. However, the approximations of generator currents in the first formulation are presumably less accurate than that of the second formulation, which may indicate that, if the distribution system operator were to implement the generation dispatch of both simulations, the actual power supply costs associated with the first solution could be higher than those of the second solution. In order to verify if this is in fact the case, the optimal decisions of both simulations (network topology and active power output of generators) will be used as fixed inputs for simulations of the

power flow in the distribution system. Due to the fact that both topologies are now meshed, it is no longer possible to use a *backward-forward power flow algorithm* for the simulations. Thus, both operating points will be simulated with help of the NL-ACOPF software OptFlow [87]. For these simulations, the active and reactive power output of all generators in the network are considered as fixed, and the ACOPF is thus employed simply to obtain the solution corresponding to the operating point described by the input data – which is made to obtain the amount of active power imported from the external system (the active power infeed at the slack bus). With these results at hand, the value of the *actual system operation costs* have been calculated, analogously to what has been done in subsection 5.2.3.1.

As in subsection 5.2.3.1, the operation costs obtained by the procedure described in the previous paragraph are compared to those corresponding to the optimal solution of a NL-ACOPF, also obtained with help of the software OptFlow [87]. Now, the network topologies corresponding to the switching decisions registered in Table 5.17 and Table 5.15 are used as fixed inputs for NL-ACOPF simulations, through which the optimal generator dispatch is determined (i.e., the active power outputs are now decision variables of the NL-ACOPF). The costs corresponding to the optimal decisions obtained with the NL-ACOPF will be used to benchmark the *actual system operation costs* obtained by the procedure described in the previous paragraph.

The results of the procedure described in the two previous paragraphs are indicated in Table 5.16.

Table 5.16. Comparison of solutions for application A3 with meshed operation allowed: simulation of *actual system operation costs* associated with the solutions obtained by the MILP formulation with McCormick’s envelopes and by the MILP formulation with piecewise-linear approximations, as well as the operating costs obtained with the NL-ACOPF, with the corresponding network topologies considered as fixed.

Item		Simulation of solution obtained by the MILP formulation, with NL-ACOPF	Solution with NL-ACOPF (considering network topology of Table 5.14)	Simulation of solution obtained by the MILP formulation, with NL-ACOPF	Solution with NL-ACOPF (considering network topology of Table 5.15)
		Formulation with McCormick’s envelopes		Formulation with piecewise-linear approximations	
Active power output [MW]	Gen. at bus 7	0.500	0.500	0.500	0.500
	Gen. at bus 14	0.161	0.296	0.300	0.300
	Gen. at bus 17	0.179	0.300	0.300	0.300
	Gen. at bus 29	0	0	0	0

Item	Simulation of solution obtained by the MILP formulation, with NL-ACOPF	Solution with NL-ACOPF (considering network topology of Table 5.14)	Simulation of solution obtained by the MILP formulation, with NL-ACOPF	Solution with NL-ACOPF (considering network topology of Table 5.15)
	Formulation with McCormick's envelopes		Formulation with piecewise-linear approximations	
Power imports [MW]	3.31	3.08	3.06	3.06
Actual system op. costs [\$]	475.7	479.5	477.7	477.7

The results of Table 5.16 indicate that the actual system operation costs obtained with the *formulation with piecewise-linear approximations* are lower than those obtained with the *formulation with McCormick's envelopes*. In fact, the generation dispatch obtained with the *formulation with piecewise-linear approximations* is identical to that obtained with the benchmark method (NL-ACOPF considering the network topology as fixed). The reader should notice that the execution time for the *formulation with piecewise-linear approximations* (1414.3 s) is considerably higher than the execution time for the *formulation with McCormick's envelopes* (152.0 s).

5.2.4 Application A4: minimum-cost expansion plan

For this application A4, it is assumed that the user wishes to determine the distribution system expansion plan, involving capacitor placement, reinforcements to circuits and a possible new step-down substation, such that the sum of investments costs and costs of ohmic losses is minimized.

The distribution system considered for this third application, referred to as S8, is based on the input data for test system S3. The following modifications have been made to obtain S8:

- (i) The 37 branches of the original system are considered as existing circuits, and 11 of these consist of switches;
- (ii) The load at all buses are increased to 125% of their original value;
- (iii) The voltage at the interface with the transmission system is assumed to be of 1.0 p.u. and the admissible range for the voltage magnitude of all buses in the system is set to $0.95 \text{ p. u.} \leq V_k \leq 1.05 \text{ p. u.}$;
- (iv) The following are defined as candidates for system expansion:
 - A new distribution substation;

- Circuits connecting the low-voltage bus of the new distribution substation to the existing buses 13 and 15;
- Circuits between the following existing buses (8-13), (9-21), (15-26) and (15-30);
- Capacitors connected to the existing buses 5, 11 and 31 (modeled as purely capacitive loads at the auxiliary buses 805, 811 and 831, which are connected to the existing buses via low-voltage fictitious circuits).

The network topology of S8 indicated in Figure 5.3.

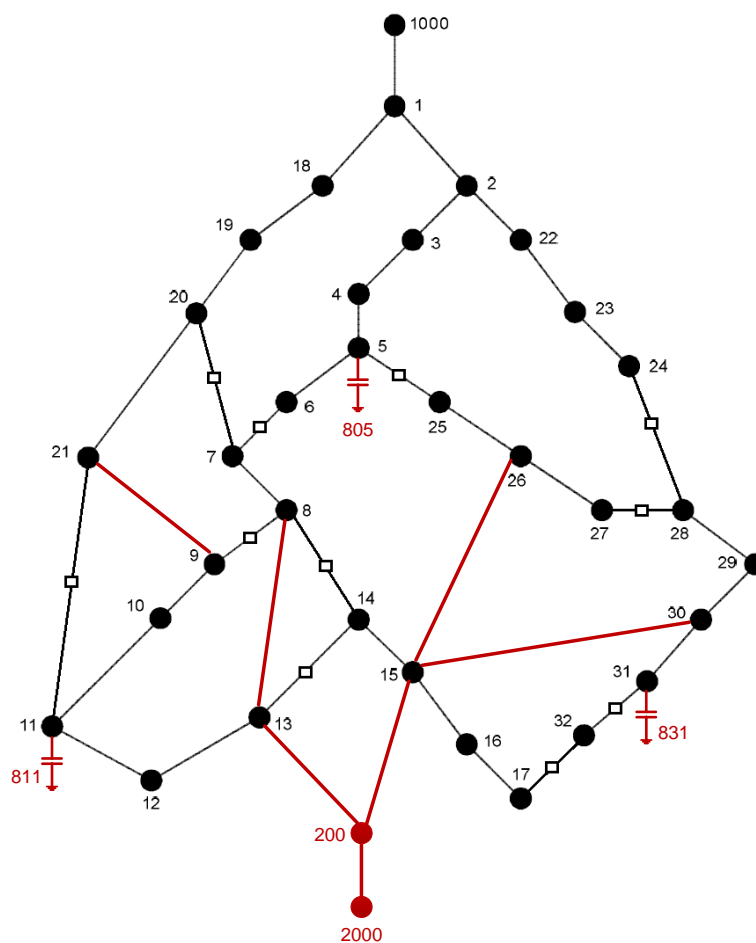


Figure 5.3: Schematic diagram of system S8. Switchable branches are indicated with a square, candidate facilities are marked in red. Adapted from [49].

It is assumed that the system operating point evaluated by the user is representative of a year, and that the optimal expansion plan is that which minimizes the sum of the annualized investment costs of the facilities and the costs of losses within a

year. The costs of losses are obtained simply by multiplying the losses in MW by 8760 hours (typical duration of a year), and then multiplying the result by the cost coefficient of 125 \$/MWh. Table 5.17 indicates the annualized investment costs of each candidate facility.

Table 5.17. Annualized invest costs of candidate facilities.

ID	Candidate	Representation	Annualized investment costs [\$]
1	Step-down substation and associated transformer	Circuit (2000-200)	48,000
2	Circuit between low voltage bus of new substation and existing bus 13	Circuit (200-13)	7,800
3	Circuit between low voltage bus of new substation and existing bus 15	Circuit (200-15)	7,200
4	Circuit between existing buses 8 and 13	Circuit (8-13)	9,000
5	Circuit between existing buses 9 and 21	Circuit (9-21)	8,400
6	Circuit between existing buses 15 and 26	Circuit (15-26)	12,000
7	Circuit between existing buses 15 and 30	Circuit (15-30)	11,000
8	Capacitor at bus 5	Purely reactive load at bus 805	9,000
9	Capacitor at bus 11	Purely reactive load at bus 811	9,000
10	Capacitor at bus 29	Purely reactive load at bus 829	9,000

The complete input data for system S8 are presented in detail in Appendix A (section 7.2.4).

The connectivity approach used in this application is that described in section 4.2.1.7.3 – approach (iii). This approach is employed to ensure that, if the new substation is not built, the choice to build bus 200 (to which no loads or generators connect, and that is not a slack bus) is taken solely based on the impacts of this decision on the objective function.

The following objective function is employed for the minimization of the sum of investments costs and costs of ohmic losses:

$$z^{REIN \& CAP \& LOSS} = \min \left\{ \sum_{km \in \Psi_{CD}} c_{km}^{CONST} \cdot \sigma_{km} + \sum_{k \in \Omega_{CAP}} c_k^{CAPL} \cdot (1 - \rho_k) \right. \\ \left. + c_k^{LOSS} \cdot \left[\sum_{k \in \Omega_{ITFC}} V_k^{ref} \cdot I_{g,k}^{re} - \sum_{k \in \Omega_{PCTE}} d_k^P \right] \right\} \quad (355)$$

The optimal solution to the problem is summarized in Table 5.18 and Figure 5.4.

Table 5.18. Case study results: application A4, minimization of the sum of investments costs and costs of ohmic losses.

Test system	Objective function			Reinforcements to distribution system		Switching decisions (status of switchable branches)		Execution time [s]
	Objective function [\$]	Annual losses costs [\$]	Annualized investment costs [\$]	ID	Annualized investment costs [\$]	Switched-on branches	Switched-off branches	
S8	188,568	114,568	74,000	1	48,000	(5-25), (7-20), (17-32)	(6-7), (8-9), (8-14), (13-14), (27-28), (31-32), (11-21), (24-28)	134.6
				2	7,800			
				3	7,200			
				7	11,000			

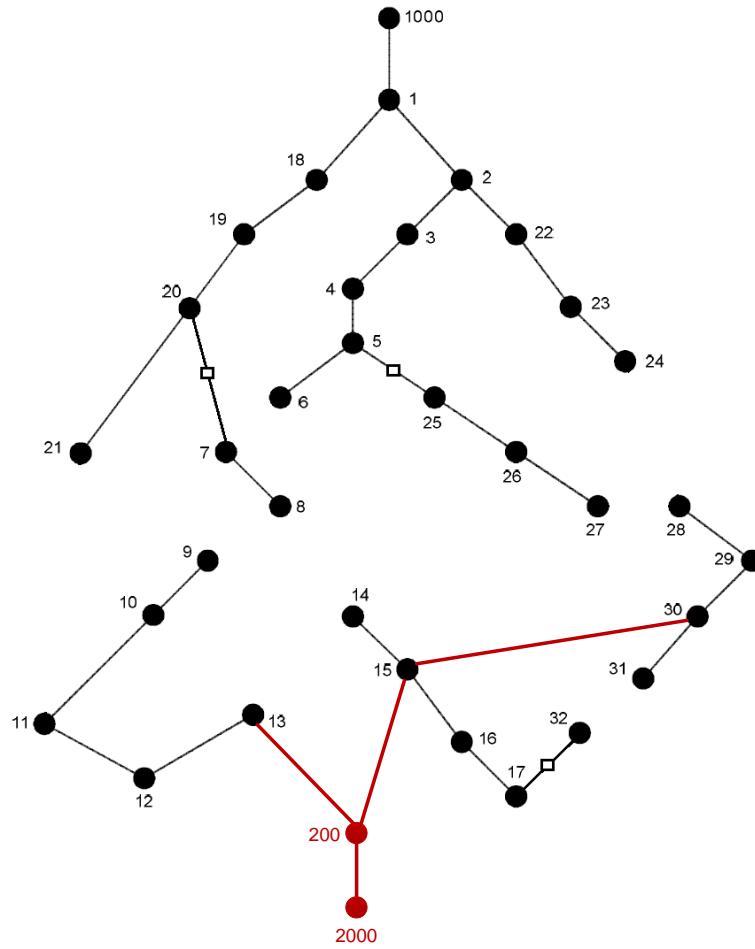


Figure 5.4: Optimal distribution system expansion plan, application A4, system S8. Adapted from [49].

The bus voltages corresponding to the optimal solution of the MILP reformulation of the ACOPF (the system topology shown in Figure 5.4) are indicated in Table 5.19. In order to quantify the approximation errors of the bus voltage magnitudes, the operating point corresponding to the optimal solution of the MILP problem has been used as input data for a simulation using the *backward-forward* load flow algorithm [67], analogously to what has been done in section 5.2.2 to quantify the approximation errors of bus voltages. The results of this comparison are shown in Table 5.19. The approximation errors range from 0.00005% to 0.0017% for voltage magnitudes and from -0.71% to 0.40% for bus voltage angles (excluding the reference voltage bus).

Table 5.19: Bus voltage magnitude profile: solution of MILP ACOPF (MILP), simulation with backward-forward load flow (BFLF) and comparison of relative error, given by (MILP-BFLF)/BFLF.

Bus #	Voltage magnitude [p.u.]			Voltage angle [°]		
	BFLF	MILP	(MILP-BFLF)/BFLF [%]	BFLF	MILP	(MILP-BFLF)/BFLF [%]
1000	1.00000	1.00000	-	0.00000	0.00000	-
1	0.99785	0.99785	0.00009%	-0.00274	-0.00272	-0.7050%
2	0.98983	0.98982	0.0006%	-0.01202	-0.01195	-0.6040%
3	0.98759	0.98758	0.0007%	-0.01453	-0.01445	-0.5773%
4	0.98573	0.98572	0.0008%	-0.02046	-0.02037	-0.4570%
5	0.98180	0.98179	0.0011%	-0.09219	-0.09210	-0.0893%
6	0.98101	0.98100	0.0011%	-0.14092	-0.14086	-0.0439%
7	0.97953	0.97952	0.0013%	-0.37633	-0.37610	-0.0612%
8	0.97892	0.97891	0.0013%	-0.38742	-0.38719	-0.0618%
9	0.99147	0.99147	0.0003%	-0.08112	-0.08109	-0.0449%
10	0.99157	0.99157	0.0003%	-0.08114	-0.08110	-0.0439%
11	0.99193	0.99193	0.0003%	-0.08374	-0.08370	-0.0384%
12	0.99461	0.99461	0.0002%	-0.05392	-0.05390	-0.0419%
13	0.99624	0.99624	0.0002%	-0.01083	-0.01081	-0.2295%
14	0.99030	0.99029	0.0005%	0.03043	0.03055	0.4003%
15	0.99069	0.99069	0.0004%	0.04193	0.04205	0.3012%
16	0.98719	0.98718	0.0006%	-0.06437	-0.06428	-0.1449%
17	0.98596	0.98595	0.0007%	-0.07701	-0.07692	-0.1223%
18	0.99669	0.99669	0.0002%	-0.02570	-0.02567	-0.1172%
19	0.98778	0.98777	0.0007%	-0.18765	-0.18752	-0.0726%
20	0.98558	0.98558	0.0008%	-0.24672	-0.24657	-0.0629%
21	0.98478	0.98477	0.0009%	-0.27250	-0.27235	-0.0583%
22	0.98537	0.98536	0.0009%	-0.05019	-0.05010	-0.1803%
23	0.97706	0.97705	0.0013%	-0.15993	-0.15983	-0.0663%
24	0.97292	0.97290	0.0017%	-0.21411	-0.21400	-0.0545%
25	0.98145	0.98144	0.0011%	-0.09423	-0.09414	-0.0885%
26	0.98113	0.98112	0.0011%	-0.09635	-0.09626	-0.0876%
27	0.98047	0.98046	0.0012%	-0.11253	-0.11245	-0.0711%
28	0.97142	0.97141	0.0014%	0.31344	0.31402	0.1838%
29	0.97206	0.97205	0.0014%	0.31131	0.31188	0.1808%
30	0.97974	0.97973	0.0010%	0.14960	0.14993	0.2221%
31	0.97893	0.97892	0.0010%	0.12866	0.12898	0.2513%
32	0.98556	0.98555	0.0008%	-0.08161	-0.08152	-0.1155%
2000	1.00000	1.00000	-	0.00000	0.00000	-
200	0.99860	0.99860	0.00005%	0.01333	0.01335	0.1546%

5.2.5 Application A5: voltage control at distribution substation to minimize active power requirements

So far, all of the objective functions (or modules for objective functions) presented in section 4.2.2 have been used in at least one application – except for the

application of circuit switching costs (section 4.2.2.8), which is essentially analogous to the minimization of the costs of construction of new circuits, as costs are associated with the modification of the status of a given circuit.

This section 5.2.5 deals with an application that was not directly mentioned in section 4.2.2, due to its very particular nature. It is assumed that the country in which the distribution system is located is experiencing problems with the security of energy supply, and having difficulties in meeting the total energy demand. Among the measurements under consideration for reducing the demand for electrical energy in this fictitious country is the operation of distribution systems at voltage magnitudes lower than the usual admissible range, in order to forcefully reduce the overall active power requirements. The operations planner of a given distribution utility is thus required to execute a study to indicate the optimal setpoint of the voltage magnitude at the interface of its system with the transmission network, such that the overall power requirements of its system will be minimized.

It is assumed that the lower bound of the *usual* admissible range for the voltage magnitudes at all buses in the system, 0.95 p.u., is to be substituted by 0.8 p.u. – with the value of 0.8 p.u. assumed to be the lowest possible voltage at which is ensured that no damage is inflicted to any equipment (distribution facilities or consumer's loads).

It is worth mentioning that minimizing the overall power requirements does not necessarily mean operating at the lowest possible voltage, mainly due to the fact that the magnitude of the current demanded by loads of the constant-power type will increase as voltage decreases, leading to an increase in the ohmic losses within the distribution system. Thus, the actual optimal operating voltage will depend on the nature, location and magnitude of the loads in the distribution system.

The distribution system considered for this fifth application, referred to as S9, is obtained by the modification of the input data for test system S3. The following modifications have been made to obtain S9:

- (i) Only 11 of the 37 branches in the system are considered to be switchable;
- (ii) The voltage magnitude at the interface with the transmission system is assumed to be fully controllable.
- (iii) The admissible range for the voltage magnitude of all buses in the system is set to $0.8 \text{ p. u.} \leq V_k \leq 1.05 \text{ p. u.}$;

- (iv) All loads in the original system S3 were of the constant-power type. In system S9, it is assumed that, from the 32 loads within the distribution system, 11 are of the constant-power type, 7 of the constant-current type, and 14 of the constant-impedance type. The nominal power associated with each of the load types is respectively of 1.19 MW, 0.71 MW and 1.815 MW – i.e., the percentage of the loads of the constant-power, constant-current and constant-impedance type are of 32%, 19% and 49% of the total load in S9.

The input data for system S9, used for application A5, are presented in detail in Appendix A (section 7.2.5). The network topology of S9 is identical to that indicated in Figure 5.2.

As there are no generators in system S9, minimizing the overall power requirements equals minimizing the total power imports at the interface with the transmission system. Thus, one of the alternative formulations for the objective function of section 4.2.2.4 may be used, with the cost coefficient c_k^{IMPORT} set to unity. Obviously, as the voltage magnitude at the interface with the transmission system is a decision variable in the problem, it is necessary to approximate the product $V_k \cdot I_{g,k}^{re}$. For that, a piecewise-linear approximation with the use of SOS2, which has been presented in section 4.2.2.4.2, will be used. The procedure for the determination of the evaluation points and evaluation values corresponds to that indicated in section 4.3.2.3, with $\underline{p}_k = 0$ MW, $\bar{p}_k = 5$ MW, $E^V = 5$ and $E^{Ig} = 11$.

The objective function employed for application A5 is:

$$z^{IMPR} = \min\{\sum_{k \in \Omega_{ITFC}} P_k\} \quad (356)$$

The optimal solution for this problem is summarized in Table 5.20.

Table 5.20. Case study results: application A5, minimization of system power requirements.

Test system	System power requirements [MW] (objective function)	Optimal voltage at the interface with transmission system [p.u.]	Switching decisions (status of switchable branches)		Execution time [s]
			Switched-on branches	Switched-off branches	
S9	3.113	0.8528	(5-25), (7-20), (8-14), (11-21), (17-32), (24-28)	(6-7), (13-14), (8-9), (27-28) (31-32)	16.53

Figure 5.5 indicates the voltage profile across the distribution network at the optimal solution of the MILP problem.

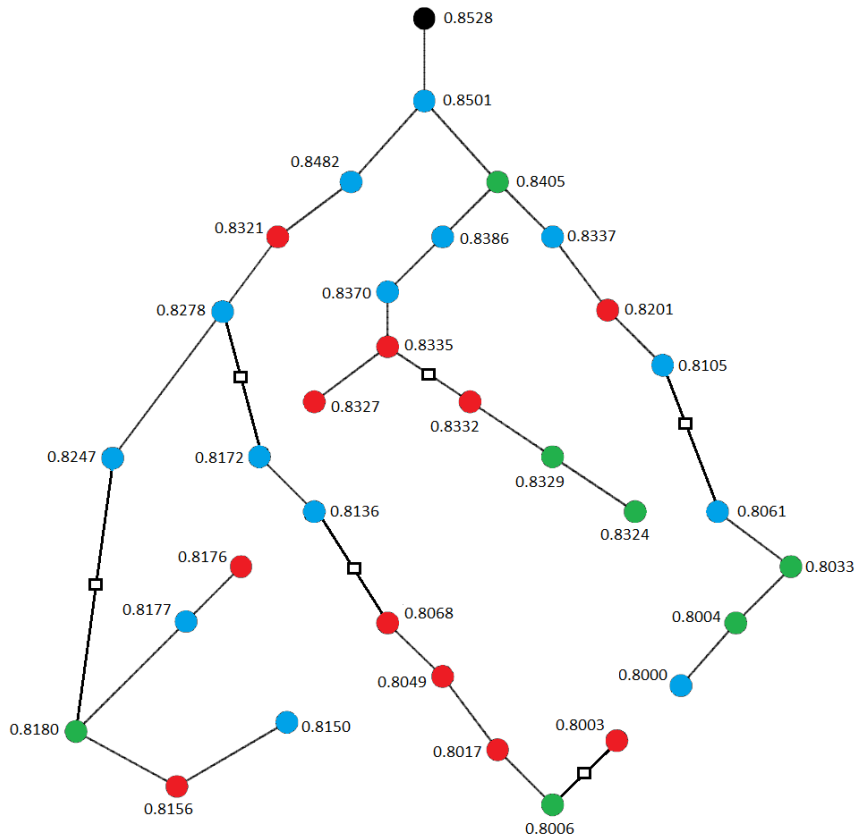


Figure 5.5: Voltage magnitude profile at the optimal solution. Buses with loads of the *constant-power*, *constant-current* and *constant-impedance* type are marked in red, green, and blue. Adapted from [49].

Auxiliary analyses have been executed for system S9, considering situations in which all loads were considered to be of the constant-power, constant-current and constant-impedance type. Obviously, the solutions obtained for these three cases differ

from that obtained when a combination of the three types of loads is considered – the solutions differ with respect to the optimal voltage magnitude at the interface with the transmission system, the optimal configuration of the distribution network and the value of the objective function. For the cases in which all loads are considered to be of the constant-power, constant-current and constant-impedance type, the optimal voltage magnitudes at bus 1000 (the interface with the transmission system) is respectively of 1.050 p.u., 0.8804 p.u. and 0.8450 p.u., and the associated power requirements of the distribution system are of 3.848 MW, 3.272 MW and 2.564 MW.

Before closing this section, it is worth considering one last auxiliary analysis for application A5. For this last auxiliary analysis, it is considered that system S9 has exactly the same composition of loads indicated in Appendix A – i.e., the composition presented in item (iv) at the beginning of this section. However, it is now considered that there are three switchable capacitors in the system, connected to buses 5, 11 and 31. Each of these switchable capacitors has a nominal rating of 200 kVAr and is modeled as a purely reactive load of the constant-impedance type at fictitious buses connected to the main network through low-impedance branches. This representation is virtually identical to that of the candidate capacitors of application A4, the difference being that no costs are associated with changing the status of the capacitors from active to inactive for the current analysis.

It is expected that the presence of switchable capacitors within the distribution network allows a better control over the voltage profile and reduces the losses by providing local reactive power resources, thus allowing a further decrease in the total active power requirements of the distribution system.

This modified version of system S9, with the addition of the abovementioned switchable capacitors, has been used for the problem of minimization of total power requirements via control of the bus voltage magnitude at the reference bus. The results of this auxiliary analysis, considering the modified version of S9, are summarized in Table 5.21. All capacitors are switched-on in the optimal solution of the problem, allowing that the voltage at the interface with the transmission system to be slightly reduced, and also slightly reducing the total power requirements of the distribution system (a reduction of 0.5%).

Table 5.21. Case study results: application A5, minimization of system power requirements, with modified system S9 (inclusion of switchable capacitors with nominal rating of 200 kVAR at buses 5, 11 and 31).

Test system	System power requirements [MW] (objective function)	Optimal voltage at the interface with the transmission system [p.u.]	Switching decisions			Execution time [s]
			Status of switchable branches		Status of switchable capacitors	
			Switched-on branches	Switched-off branches	Buses with switched-on capacitors	
S9 with switchable capacitors	3.099	0.8503	(5-25), (7-20), (8-14), (11-21), (17-32), (24-28)	(6-7), (13-14), (8-9), (27-28) (31-32)	5, 11, 31	96.83

Figure 5.6 indicates the voltage profile across the distribution network for the modified version of system S9 (with the inclusion of switchable capacitors).

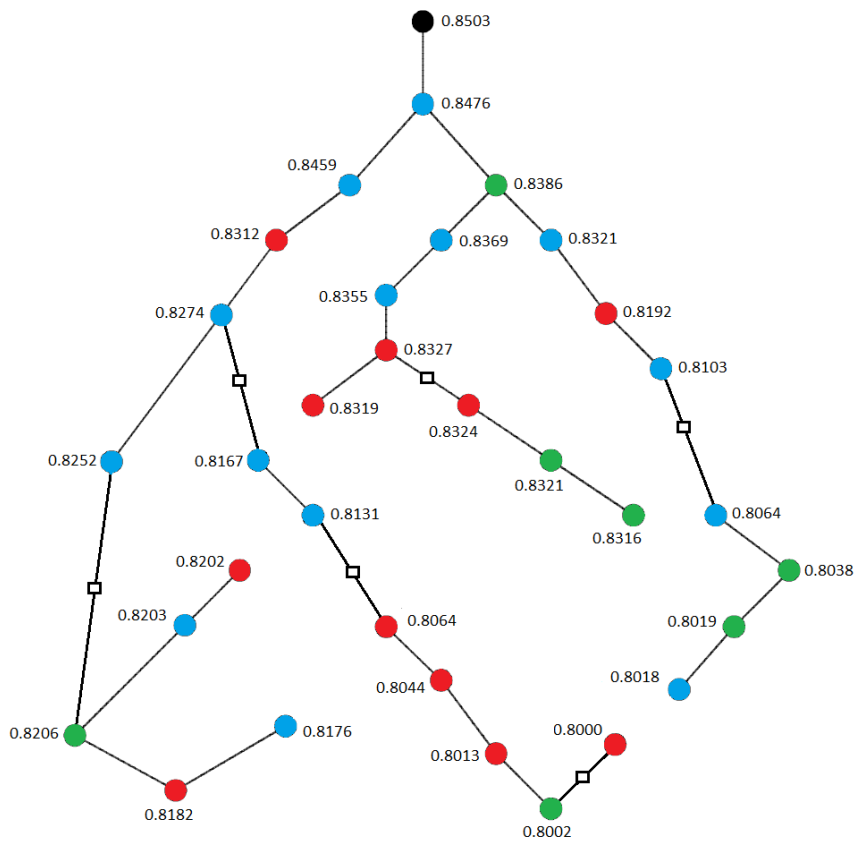


Figure 5.6: Voltage magnitude profile at the optimal solution, for modified system (with capacitors added). Buses with loads of the constant-power, constant-current and constant-impedance type are marked in red, green, and blue. Adapted from [49].

A comparison of the bus voltage profile of the solutions indicated in this section 5.2.5 with the bus voltages obtained by simulations with a backward-forward load flow

algorithm is not made at this point, due to the fact that the currently available backward-forward load flow algorithm does not support loads of the constant-current type.

6 CONCLUSIONS

In this dissertation, a MILP reformulation of the ACOPF problem for distribution systems, which allows the incorporation of discrete decisions associated with several distribution system operations and expansion planning applications, has been proposed.

The proposed formulation is based on expressing Kirchhoff's laws as a function of complex voltages and currents in rectangular coordinates – as opposed to employing a formulation based on polar coordinates and using voltages and power quantities. This modeling choice allows that particular characteristics of the distribution system are taken advantage of while formulating the problem, with the goal of conciliating accuracy and computational performance.

The choice of reformulating the ACOPF problem as MILP allows the prompt modeling of many of the discrete decisions with which distribution system operations and expansion planners are faced, such as the maneuvering of switches for network reconfiguration and the construction of facilities for system expansion. The fact that the proposed formulation simultaneously supports discrete and continuous decisions widens its applicability to a wide range of distribution system operations and expansion planning problems – and some of these have been illustrated with help of the case studies of chapter 5.

Other practical advantage of reformulating the ACOPF as a MILP is that the solution techniques for mixed-integer linear programs are notably mature, allowing the treatment of large-scale optimization problems with robustness and speed. These techniques are readily available in a number of commercial-grade solvers. The possibility of using commercial solvers is an attractive feature for industry applications, as it essentially translates into guarantees of longevity, maintainability and prevention of obsolescence of the solver that underlies a decision support system.

The linearization and convexification techniques presented in chapter 3 have been employed to reformulate the original non-convex, non-linear ACOPF problem as a mixed-integer linear program. With the exception of the approximation of bilinear products with McCormick's envelope, which have been employed to reformulate the products of decision variables within the equations for the current injections of

generators, the proposed linearization and convexification techniques allow the user to obtain approximations of arbitrary accuracy – i.e., the user is able control the accuracy of these approximations while formulating the problem. In this dissertation, reference has been made to the possibility of employing piecewise-linear approximations based on SOS2 to reformulate products of decision variables – and in fact this technique has been employed in the equations of subsection 4.3.1.4.4. The technique has also been employed for reformulating the bilinear products found in the constraints for generator current injections, in the alternative formulation presented in Appendix C (chapter 9), which have been used in the case study of subsection 5.2.3. Employing this technique in fact allows the user to control the accuracy of all approximations used in the reformulation of the ACOPF for distribution systems.

The use of these linearization and convexification techniques requires the definition of the following input parameters: *disjunctive constants* for the definition of disjunctive constraints, *evaluation points* and *evaluated values* for the definition of piecewise-linear approximations with SOS2, and *upper and lower bounds* for the continuous variables whose product is modeled via McCormick’s envelope. In this dissertation, particular characteristics of the distribution system (mainly the fact that bus voltage angles vary within narrow intervals around the reference angle, due to the high R/X ratios and the typical power factors of loads within the system) have been explored to obtain a *tight* definition of the abovementioned parameters. This means that the parameters are defined in such a way that allows the correct representation of the problem, while seeking a satisfactory trade-off between approximation accuracy and computational performance.

In chapter 5, the proposed MILP reformulation of the ACOPF has been benchmarked against a brute-force, exhaustive search solution method, for the problem of network reconfiguration with the goal of minimizing ohmic losses. The problem of network reconfiguration has been chosen because it involves exclusively *binary decisions*. While this was necessary to allow a construction of a brute force algorithm against which the MILP reformulation could be benchmarked, it is worth mentioning that many of the features of the proposed formulation are not put into service while solving the network reconfiguration problems of section 5.1. For instance, one relevant feature of the proposed MILP formulation is that it is able to support discrete *and*

continuous decisions, and this feature is clearly not thoroughly explored in the network reconfiguration problems of section 5.1.

For all four test systems used in the benchmarking process mentioned above, the optimal network configuration obtained with the proposed MILP reformulation of the ACOPF perfectly matched that obtained with the brute-force method, meaning that the same switching decisions have been made with both methods. This effectively means that, if the optimal decisions obtained with both methods were implemented, the same ohmic losses in distribution network would be obtained – indicating that the *actual value* of the objective function obtained with the MILP formulation and the brute-force algorithm is identical, for all case studies.

The *approximated numerical value* of the objective function obtained with the MILP reformulation (i.e., the approximated numerical value corresponding to the solution of the mixed-integer program, and not the *actual value* that would be obtained by implementing the solution) is also similar to that obtained with the exhaustive search method. In fact, the relative differences between the *numerical value* of the optimal ohmic losses obtained with the MILP reformulation and with the exhaustive search method varied from -0.8% for the system with 11 branches to 6% for the system with 134 branches.

For systems with intermediate dimensions and a comparatively larger number of switches, the performance of the proposed MILP reformulation has been superior to that of the exhaustive search method – e.g., for system S3, in which there are 2^{26} possible network configurations to be analyzed, the solution time with the MILP reformulation was only 0.95% of the solution time with the brute force method. However, for larger systems with a comparatively smaller number of switches, the brute force method has outperformed the MILP reformulation – e.g., for system S4, with 134 branches and 2^{16} possible network configurations, the exhaustive search method has been 5.8% faster than the MILP reformulation in finding the optimal decision. While analyzing this last result, the reader should keep in mind that, if the problem under consideration in the benchmarking analyses required the support to continuous decisions, the brute force algorithm could simply not be used, while the MILP reformulation would still apply.

A number of possible applications of the proposed MILP reformulation of the ACOPF problem have been illustrated by the case studies of section 5.2. The case studies of this section referred to the minimization of costs of load shedding, generation

curtailment, variable costs of generation, costs of power imports and costs of reinforcements to the distribution system. This list does not aim at being exhaustive with respect to the possible applications of the proposed formulation, but solely at indicating its flexibility. Selected numerical results of these case studies have also been benchmarked against results obtained by simulating the solution of the mixed-integer program with a backward-forward load flow method, and the comparison also pointed to a satisfactory accuracy of the proposed MILP reformulation. All case studies have been built upon test systems obtained by modification of the data originally proposed in [64] (a system with 33 buses and 37 branches), with slight modifications in the number of elements in the network for certain applications. The execution times for the case studies of section 5.2 ranged from 11.03 s to 1602.8 s. The latter execution time has been obtained for an application with 11 switchable circuits and 20 curtailable generators – totalizing 2^{31} possible combinations of these binary variables, which model operations planning decisions.

The results of the case studies of chapter 5 suggest that the proposed MILP reformulation of the ACOPF for distribution systems meets the goals of accurately capturing the non-linear behavior of the original problem and leading to solutions of good quality, while being flexible enough to support a wide range of applications. It is worth mentioning that the MILP reformulation of the ACOPF has been coded and executed with FICO Xpress Mosel ® Version 3.2.2 – a commercial-grade solver, which brings about the practical advantages mentioned at the beginning of this chapter. The solution times obtained for the applications may be classified as satisfactory, though there seems to be room for improvement – as indicated in the first paragraph of the following section.

6.1 Suggested topics for future work

Techniques for improving the computational performance of the proposed MILP reformulation of the ACOPF are among the suggested topics for future work. The reader will recall that the piecewise-linear approximations of functions of two variables employed in the current formulation are based in arranging the set of *evaluation points* at the vertices of a *rectangular* grid, which is superimposed to the function domain. It is likely that the procedures for the construction of piecewise-linear approximations based

on constructing *triangular grids* of evaluation points will lead to enhanced computational performance, as suggested in [61], [80]. Other possible approaches to be investigated, eventually in combination with the one described above, include: (a) using the technique described in [86] to reduce the number of binary variables necessary to implement SOS2-based piecewise linear approximations¹³; and (b) employing linearization and convexification techniques other than those described in this dissertation. In general terms, techniques that ensure that the choice of evaluation points is optimal either with respect to accuracy (e.g., minimizing the maximum approximation error while keeping the number of points below a certain threshold) or computational performance (e.g., minimizing the number of points while keeping the maximum approximation error below a certain threshold) are suggested as topics for future work.

The improvement of the computational is an important research topic also in order to allow the practical use of the proposed formulation in problems in which multiple operating conditions have to be evaluated – e.g., in stochastic and multi-stage problems.

The expansion of the proposed MILP reformulation of the ACOPF problem to unbalanced three-phase distribution systems may also be an interesting topic for future work, taking into account that phase unbalance is an important phenomenon in many real distribution systems. Future work may also include modeling of other equipment relevant for distribution systems, such as voltage regulators.

¹³ Reference [86] basically suggests using a special encoding procedure for the definition intervals of the piecewise linear function, allowing the use of a logarithmic number of binary variables.

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7 APPENDIX A: INPUT DATA FOR CASE STUDIES

The input data for the case studies of chapter 5 are presented in the following subsections, in tabular form. The apparent power base for all quantities expressed in per unit (p.u.) is 100 MVA. The nomenclature presented in chapters 2 and 4 is used for ensuring a succinct presentation of data.

7.1 Input data for distribution systems used in section 5.1

7.1.1 Test system S1

Table 7.1. Bus data: test system S1

Bus #	Sets to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.95	1.05	-5	5
5	Ω_{PCTE}		2	0.6	0.95	1.05	-5	5
4	Ω_{PCTE}		3	1.3	0.95	1.05	-5	5
3	Ω_{PCTE}		2	0.5	0.95	1.05	-5	5
2	Ω_{PCTE}		1.5	0.3	0.95	1.05	-5	5
1	Ω_{PCTE}		0.5	0.1	0.95	1.05	-5	5
101	Ω_{PCTE}		1	0.2	0.95	1.05	-5	5
102	Ω_{PCTE}		1.5	0.2	0.95	1.05	-5	5
103	Ω_{PCTE}		2.5	0.6	0.95	1.05	-5	5
104	Ω_{PCTE}		3	0.4	0.95	1.05	-5	5
105	Ω_{PCTE}		2.5	0.9	0.95	1.05	-5	5
2000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.95	1.05	-5	5

Table 7.2. Branch data: test system S1

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1000	5	Ψ_{SW}	0.075	0.1	0.25
5	4	Ψ_{SW}	0.08	0.11	0.25
4	3	Ψ_{SW}	0.09	0.12	0.25
3	2	Ψ_{SW}	0.04	0.04	0.25
2	1	Ψ_{SW}	0.03	0.03	0.25
1	101	Ψ_{SW}	0.04	0.01	0.25
101	102	Ψ_{SW}	0.1	0.1	0.25
102	103	Ψ_{SW}	0.11	0.11	0.25
103	104	Ψ_{SW}	0.09	0.12	0.25
104	105	Ψ_{SW}	0.055	0.11	0.25
105	2000	Ψ_{SW}	0.1	0.1	0.25

7.1.2 Test system S2

Table 7.3. Bus data: test system S2

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.95	1.05	-5	5
2	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.95	1.05	-5	5
3	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.95	1.05	-5	5
4	Ω_{PCTE}		2	1.6	0.95	1.05	-5	5
5	Ω_{PCTE}		3	1.5	0.95	1.05	-5	5
105	Ω_{ZCTE}		0	-1.1	0.95	1.05	-5	5
6	Ω_{PCTE}		2	0.8	0.95	1.05	-5	5
106	Ω_{ZCTE}		0	-1.2	0.95	1.05	-5	5
7	Ω_{PCTE}		1.5	1.2	0.95	1.05	-5	5
8	Ω_{PCTE}		4	2.7	0.95	1.05	-5	5
9	Ω_{PCTE}		5	3	0.95	1.05	-5	5
109	Ω_{ZCTE}		0	-1.2	0.95	1.05	-5	5
10	Ω_{PCTE}		1	0.9	0.95	1.05	-5	5
11	Ω_{PCTE}		0.6	0.1	0.95	1.05	-5	5
111	Ω_{ZCTE}		0	-0.6	0.95	1.05	-5	5
12	Ω_{PCTE}		4.5	2	0.95	1.05	-5	5
112	Ω_{ZCTE}		0	-3.7	0.95	1.05	-5	5
13	Ω_{PCTE}		1	0.9	0.95	1.05	-5	5
14	Ω_{PCTE}		1	0.7	0.95	1.05	-5	5
114	Ω_{ZCTE}		0	-1.8	0.95	1.05	-5	5
15	Ω_{PCTE}		1	0.9	0.95	1.05	-5	5
16	Ω_{PCTE}		2.4	1	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
116	Ω_{ZCTE}		0	-1.8	0.95	1.05	-5	5

Table 7.4. Branch data: test system S2

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1	4	Ψ_{SW}	0.075	0.1	0.35
4	5	Ψ_{SW}	0.08	0.11	0.35
4	6	Ψ_{SW}	0.09	0.18	0.35
6	7	Ψ_{SW}	0.04	0.04	0.35
2	8	Ψ_{SW}	0.11	0.11	0.35
8	9	Ψ_{SW}	0.08	0.11	0.35
8	10	Ψ_{SW}	0.11	0.11	0.35
9	11	Ψ_{SW}	0.11	0.11	0.35
9	12	Ψ_{SW}	0.08	0.11	0.35
3	13	Ψ_{SW}	0.11	0.11	0.35
13	14	Ψ_{SW}	0.09	0.12	0.35
13	15	Ψ_{SW}	0.08	0.11	0.35
15	16	Ψ_{SW}	0.04	0.04	0.35
5	11	Ψ_{SW}	0.04	0.04	0.35
10	14	Ψ_{SW}	0.04	0.04	0.35
7	16	Ψ_{SW}	0.09	0.12	0.35
5	105	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05
6	106	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05
9	109	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
11	111	$\{\Psi_C \setminus \Psi_{sw}\}$	0	0.001	0.05
12	112	$\{\Psi_C \setminus \Psi_{sw}\}$	0	0.001	0.05
14	114	$\{\Psi_C \setminus \Psi_{sw}\}$	0	0.001	0.05
16	116	$\{\Psi_C \setminus \Psi_{sw}\}$	0	0.001	0.05

7.1.3 Test system S3

Table 7.5. Bus data: test system S3

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.90	1.10	-5	5
1	Ω_{PCTE}		0.1	0.06	0.90	1.10	-5	5
2	Ω_{PCTE}		0.09	0.04	0.90	1.10	-5	5
3	Ω_{PCTE}		0.12	0.08	0.90	1.10	-5	5
4	Ω_{PCTE}		0.06	0.03	0.90	1.10	-5	5
5	Ω_{PCTE}		0.06	0.02	0.90	1.10	-5	5
6	Ω_{PCTE}		0.2	0.1	0.90	1.10	-5	5
7	Ω_{PCTE}		0.2	0.1	0.90	1.10	-5	5
8	Ω_{PCTE}		0.06	0.02	0.90	1.10	-5	5
9	Ω_{PCTE}		0.06	0.02	0.90	1.10	-5	5
10	Ω_{PCTE}		0.045	0.03	0.90	1.10	-5	5
11	Ω_{PCTE}		0.06	0.035	0.90	1.10	-5	5
12	Ω_{PCTE}		0.06	0.035	0.90	1.10	-5	5
13	Ω_{PCTE}		0.12	0.08	0.90	1.10	-5	5
14	Ω_{PCTE}		0.06	0.01	0.90	1.10	-5	5
15	Ω_{PCTE}		0.06	0.02	0.90	1.10	-5	5
16	Ω_{PCTE}		0.06	0.02	0.90	1.10	-5	5
17	Ω_{PCTE}		0.09	0.04	0.90	1.10	-5	5
18	Ω_{PCTE}		0.09	0.04	0.90	1.10	-5	5
19	Ω_{PCTE}		0.09	0.04	0.90	1.10	-5	5
20	Ω_{PCTE}		0.09	0.04	0.90	1.10	-5	5
21	Ω_{PCTE}		0.09	0.04	0.90	1.10	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
22	Ω_{PCTE}		0.09	0.05	0.90	1.10	-5	5
23	Ω_{PCTE}		0.42	0.2	0.90	1.10	-5	5
24	Ω_{PCTE}		0.42	0.2	0.90	1.10	-5	5
25	Ω_{PCTE}		0.06	0.025	0.90	1.10	-5	5
26	Ω_{PCTE}		0.06	0.025	0.90	1.10	-5	5
27	Ω_{PCTE}		0.06	0.02	0.90	1.10	-5	5
28	Ω_{PCTE}		0.12	0.07	0.90	1.10	-5	5
29	Ω_{PCTE}		0.2	0.6	0.90	1.10	-5	5
30	Ω_{PCTE}		0.15	0.07	0.90	1.10	-5	5
31	Ω_{PCTE}		0.21	0.1	0.90	1.10	-5	5
32	Ω_{PCTE}		0.06	0.04	0.90	1.10	-5	5

Table 7.6. Branch data: test system S3

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1000	1	$\{\Psi_C \setminus \Psi_{SW}\}$	0.05753	0.02932	0.05
1	2	$\{\Psi_C \setminus \Psi_{SW}\}$	0.3076	0.15667	0.05
2	3	$\{\Psi_C \setminus \Psi_{SW}\}$	0.22836	0.1163	0.05
3	4	$\{\Psi_C \setminus \Psi_{SW}\}$	0.23778	0.1211	0.05
4	5	$\{\Psi_C \setminus \Psi_{SW}\}$	0.51099	0.44112	0.05
5	6	Ψ_{SW}	0.1168	0.38608	0.05
6	7	Ψ_{SW}	0.44386	0.14668	0.05
7	8	Ψ_{SW}	0.64264	0.4617	0.05
8	9	Ψ_{SW}	0.65138	0.4617	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
9	10	Ψ_{SW}	0.12266	0.04056	0.05
10	11	Ψ_{SW}	0.2336	0.07724	0.05
11	12	Ψ_{SW}	0.91592	0.72063	0.05
12	13	Ψ_{SW}	0.33792	0.4448	0.05
13	14	Ψ_{SW}	0.36874	0.32818	0.05
14	15	Ψ_{SW}	0.46564	0.34004	0.05
15	16	Ψ_{SW}	0.80424	1.07378	0.05
16	17	Ψ_{SW}	0.45671	0.35813	0.05
1	18	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10232	0.09764	0.05
18	19	$\{\Psi_C \setminus \Psi_{SW}\}$	0.93851	0.84567	0.05
19	20	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2555	0.29849	0.05
20	21	Ψ_{SW}	0.4423	0.58481	0.05
2	22	$\{\Psi_C \setminus \Psi_{SW}\}$	0.28152	0.19236	0.05
22	23	$\{\Psi_C \setminus \Psi_{SW}\}$	0.56028	0.44243	0.05
23	24	$\{\Psi_C \setminus \Psi_{SW}\}$	0.55904	0.43743	0.05
5	25	Ψ_{SW}	0.12666	0.06451	0.05
25	26	Ψ_{SW}	0.17732	0.09028	0.05
26	27	Ψ_{SW}	0.66074	0.58256	0.05
27	28	Ψ_{SW}	0.50176	0.43712	0.05
28	29	Ψ_{SW}	0.31664	0.16128	0.05
29	30	Ψ_{SW}	0.60795	0.60084	0.05
30	31	Ψ_{SW}	0.19373	0.2258	0.05
31	32	Ψ_{SW}	0.21276	0.33081	0.05
7	20	Ψ_{SW}	1.24785	1.24785	0.05
8	14	Ψ_{SW}	1.24785	1.24785	0.05
11	21	Ψ_{SW}	1.24785	1.24785	0.05
17	32	Ψ_{SW}	0.31196	0.31196	0.05
24	28	Ψ_{SW}	0.31196	0.31196	0.05

7.1.4 Test system S4

Table 7.7. Bus data: test system S4

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
150	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0	0.95	1.05	-5	5
1	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
2	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
3	Ω_B		0	0	0.95	1.05	-5	5
4	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
5	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
6	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
7	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
8	Ω_B		0	0	0.95	1.05	-5	5
9	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
10	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
11	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
12	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
13	Ω_B		0	0	0.95	1.05	-5	5
14	Ω_B		0	0	0.95	1.05	-5	5
15	Ω_B		0	0	0.95	1.05	-5	5
16	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
17	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
18	Ω_B		0	0	0.95	1.05	-5	5
19	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
20	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
21	Ω_B		0	0	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
22	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
23	Ω_B		0	0	0.95	1.05	-5	5
24	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
25	Ω_B		0	0	0.95	1.05	-5	5
26	Ω_B		0	0	0.95	1.05	-5	5
27	Ω_B		0	0	0.95	1.05	-5	5
28	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
29	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
30	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
31	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
32	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
33	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
34	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
35	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
36	Ω_B		0	0	0.95	1.05	-5	5
37	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
38	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
39	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
40	Ω_B		0	0	0.95	1.05	-5	5
41	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
42	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
43	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
44	Ω_B		0	0	0.95	1.05	-5	5
45	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
46	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
47	Ω_{ZCTE}		0.035	0.025	0.95	1.05	-5	5
48	Ω_{ZCTE}		0.07	0.05	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
49	Ω_{PCTE}		0.0466667	0.0316667	0.95	1.05	-5	5
50	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
51	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
52	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
53	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
54	Ω_B		0	0	0.95	1.05	-5	5
55	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
56	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
57	Ω_B		0	0	0.95	1.05	-5	5
58	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
59	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
60	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
61	Ω_B		0	0	0.95	1.05	-5	5
62	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
63	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
64	Ω_{ZCTE}		0.075	0.035	0.95	1.05	-5	5
65	Ω_{ZCTE}		0.0466667	0.0333333	0.95	1.05	-5	5
66	Ω_{PCTE}		0.075	0.035	0.95	1.05	-5	5
67	Ω_B		0	0	0.95	1.05	-5	5
68	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
69	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
70	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
71	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
72	Ω_B		0	0	0.95	1.05	-5	5
73	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
74	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
75	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
76	Ω_{ZCTE}		0.0816667	0.06	0.95	1.05	-5	5
77	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
78	Ω_B		0	0	0.95	1.05	-5	5
79	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
80	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
81	Ω_B		0	0	0.95	1.05	-5	5
82	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
83	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
84	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
85	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
86	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
87	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
88	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
89	Ω_B		0	0	0.95	1.05	-5	5
90	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
91	Ω_B		0	0	0.95	1.05	-5	5
92	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
93	Ω_B		0	0	0.95	1.05	-5	5
94	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
95	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
96	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
97	Ω_B		0	0	0.95	1.05	-5	5
98	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
99	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
100	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
101	Ω_B		0	0	0.95	1.05	-5	5
102	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
103	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
104	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
105	Ω_B		0	0	0.95	1.05	-5	5
106	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
107	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
108	Ω_B		0	0	0.95	1.05	-5	5
109	Ω_{PCTE}		0.04	0.02	0.95	1.05	-5	5
110	Ω_B		0	0	0.95	1.05	-5	5
111	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
112	Ω_{ZCTE}		0.02	0.01	0.95	1.05	-5	5
113	Ω_{ZCTE}		0.04	0.02	0.95	1.05	-5	5
114	Ω_{PCTE}		0.02	0.01	0.95	1.05	-5	5
135	Ω_B		0	0	0.95	1.05	-5	5
149	Ω_B		0	0	0.95	1.05	-5	5
151	Ω_B		0	0	0.95	1.05	-5	5
152	Ω_B		0	0	0.95	1.05	-5	5
160	Ω_B		0	0	0.95	1.05	-5	5
197	Ω_B		0	0	0.95	1.05	-5	5
250	Ω_B		0	0	0.95	1.05	-5	5
251	Ω_B		0	0	0.95	1.05	-5	5
300	Ω_B		0	0	0.95	1.05	-5	5
350	Ω_B		0	0	0.95	1.05	-5	5
450	Ω_B		0	0	0.95	1.05	-5	5
451	Ω_B		0	0	0.95	1.05	-5	5
610	Ω_B		0	0	0.95	1.05	-5	5
883	Ω_{ZCTE}		0	-0.2	0.95	1.05	-5	5
888	Ω_{ZCTE}		0	-0.05	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
890	Ω_{ZCTE}		0	-0.05	0.95	1.05	-5	5
892	Ω_{ZCTE}		0	-0.05	0.95	1.05	-5	5

Table 7.8. Branch data: test system S4

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1	2	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08486	0.08603	0.05
1	3	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
1	7	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10049	0.20587	0.05
3	4	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09698	0.09831	0.05
3	5	$\{\Psi_C \setminus \Psi_{SW}\}$	0.15759	0.15976	0.05
5	6	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
7	8	$\{\Psi_C \setminus \Psi_{SW}\}$	0.06699	0.13725	0.05
8	12	$\{\Psi_C \setminus \Psi_{SW}\}$	0.1091	0.1106	0.05
8	9	$\{\Psi_C \setminus \Psi_{SW}\}$	0.1091	0.1106	0.05
8	13	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10049	0.20587	0.05
9	14	$\{\Psi_C \setminus \Psi_{SW}\}$	0.20608	0.20892	0.05
13	34	$\{\Psi_C \setminus \Psi_{SW}\}$	0.07273	0.07374	0.05
13	18	$\{\Psi_C \setminus \Psi_{SW}\}$	0.27634	0.56614	0.05
14	11	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
14	10	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
15	16	$\{\Psi_C \setminus \Psi_{SW}\}$	0.18184	0.18434	0.05
15	17	$\{\Psi_C \setminus \Psi_{SW}\}$	0.16971	0.17205	0.05
18	19	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
18	21	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10049	0.20587	0.05
19	20	$\{\Psi_C \setminus \Psi_{SW}\}$	0.15759	0.15976	0.05
21	22	$\{\Psi_C \setminus \Psi_{SW}\}$	0.25457	0.25808	0.05
21	23	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
23	24	$\{\Psi_C \setminus \Psi_{SW}\}$	0.26669	0.27037	0.05
23	25	Ψ_{SW}	0.09211	0.18871	0.05
25	26	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09775	0.22449	0.05
25	28	$\{\Psi_C \setminus \Psi_{SW}\}$	0.06699	0.13725	0.05
26	27	$\{\Psi_C \setminus \Psi_{SW}\}$	0.07681	0.17638	0.05
26	31	$\{\Psi_C \setminus \Psi_{SW}\}$	0.1091	0.1106	0.05
27	33	$\{\Psi_C \setminus \Psi_{SW}\}$	0.24245	0.24579	0.05
28	29	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10049	0.20587	0.05
29	30	$\{\Psi_C \setminus \Psi_{SW}\}$	0.11724	0.24018	0.05
30	250	$\{\Psi_C \setminus \Psi_{SW}\}$	0.06699	0.13725	0.05
31	32	$\{\Psi_C \setminus \Psi_{SW}\}$	0.14547	0.14747	0.05
34	15	$\{\Psi_C \setminus \Psi_{SW}\}$	0.04849	0.04916	0.05
35	36	$\{\Psi_C \setminus \Psi_{SW}\}$	0.18154	0.41691	0.05
35	40	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
36	37	$\{\Psi_C \setminus \Psi_{SW}\}$	0.14547	0.14747	0.05
36	38	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
38	39	$\{\Psi_C \setminus \Psi_{SW}\}$	0.15759	0.15976	0.05
40	41	$\{\Psi_C \setminus \Psi_{SW}\}$	0.15759	0.15976	0.05
40	42	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
42	43	$\{\Psi_C \setminus \Psi_{SW}\}$	0.24245	0.24579	0.05
42	44	Ψ_{SW}	0.06699	0.13725	0.05
44	45	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09698	0.09831	0.05
44	47	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
45	46	$\{\Psi_C \setminus \Psi_{SW}\}$	0.14547	0.14747	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
47	48	$\{\Psi_C \setminus \Psi_{SW}\}$	0.05024	0.10293	0.05
47	49	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
49	50	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
50	51	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
52	53	$\{\Psi_C \setminus \Psi_{SW}\}$	0.06699	0.13725	0.05
53	54	$\{\Psi_C \setminus \Psi_{SW}\}$	0.04187	0.08578	0.05
54	55	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09211	0.18871	0.05
54	57	$\{\Psi_C \setminus \Psi_{SW}\}$	0.11724	0.24018	0.05
55	56	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09211	0.18871	0.05
57	58	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
57	60	$\{\Psi_C \setminus \Psi_{SW}\}$	0.25122	0.51467	0.05
58	59	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12122	0.12289	0.05
60	61	$\{\Psi_C \setminus \Psi_{SW}\}$	0.18423	0.37743	0.05
60	62	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2775	0.13221	0.05
62	63	$\{\Psi_C \setminus \Psi_{SW}\}$	0.19425	0.09255	0.05
63	64	$\{\Psi_C \setminus \Psi_{SW}\}$	0.3885	0.1851	0.05
64	65	$\{\Psi_C \setminus \Psi_{SW}\}$	0.47175	0.22476	0.05
65	66	$\{\Psi_C \setminus \Psi_{SW}\}$	0.36075	0.17188	0.05
67	68	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09698	0.09831	0.05
67	72	$\{\Psi_C \setminus \Psi_{SW}\}$	0.09211	0.18871	0.05
67	97	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
68	69	$\{\Psi_C \setminus \Psi_{SW}\}$	0.13335	0.13518	0.05
69	70	$\{\Psi_C \setminus \Psi_{SW}\}$	0.15759	0.15976	0.05
70	71	$\{\Psi_C \setminus \Psi_{SW}\}$	0.13335	0.13518	0.05
72	73	$\{\Psi_C \setminus \Psi_{SW}\}$	0.13335	0.13518	0.05
72	76	$\{\Psi_C \setminus \Psi_{SW}\}$	0.06699	0.13725	0.05
73	74	$\{\Psi_C \setminus \Psi_{SW}\}$	0.16971	0.17205	0.05
74	75	$\{\Psi_C \setminus \Psi_{SW}\}$	0.19396	0.19663	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
76	77	$\{\Psi_C \setminus \Psi_{sw}\}$	0.13399	0.27449	0.05
76	86	$\{\Psi_C \setminus \Psi_{sw}\}$	0.23447	0.48036	0.05
77	78	$\{\Psi_C \setminus \Psi_{sw}\}$	0.0335	0.06862	0.05
78	79	$\{\Psi_C \setminus \Psi_{sw}\}$	0.07537	0.1544	0.05
78	80	$\{\Psi_C \setminus \Psi_{sw}\}$	0.15911	0.32596	0.05
80	81	$\{\Psi_C \setminus \Psi_{sw}\}$	0.15911	0.32596	0.05
81	82	$\{\Psi_C \setminus \Psi_{sw}\}$	0.08374	0.17156	0.05
81	84	$\{\Psi_C \setminus \Psi_{sw}\}$	0.32731	0.33181	0.05
82	83	$\{\Psi_C \setminus \Psi_{sw}\}$	0.08374	0.17156	0.05
84	85	$\{\Psi_C \setminus \Psi_{sw}\}$	0.23033	0.2335	0.05
86	87	Ψ_{sw}	0.15073	0.3088	0.05
87	88	$\{\Psi_C \setminus \Psi_{sw}\}$	0.08486	0.08603	0.05
87	89	$\{\Psi_C \setminus \Psi_{sw}\}$	0.09211	0.18871	0.05
89	90	$\{\Psi_C \setminus \Psi_{sw}\}$	0.1091	0.1106	0.05
89	91	$\{\Psi_C \setminus \Psi_{sw}\}$	0.07537	0.1544	0.05
91	92	$\{\Psi_C \setminus \Psi_{sw}\}$	0.14547	0.14747	0.05
91	93	$\{\Psi_C \setminus \Psi_{sw}\}$	0.07537	0.1544	0.05
93	94	$\{\Psi_C \setminus \Psi_{sw}\}$	0.13335	0.13518	0.05
93	95	$\{\Psi_C \setminus \Psi_{sw}\}$	0.10049	0.20587	0.05
95	96	$\{\Psi_C \setminus \Psi_{sw}\}$	0.09698	0.09831	0.05
97	98	$\{\Psi_C \setminus \Psi_{sw}\}$	0.09211	0.18871	0.05
98	99	$\{\Psi_C \setminus \Psi_{sw}\}$	0.18423	0.37743	0.05
99	100	$\{\Psi_C \setminus \Psi_{sw}\}$	0.10049	0.20587	0.05
100	450	$\{\Psi_C \setminus \Psi_{sw}\}$	0.26797	0.54899	0.05
101	102	$\{\Psi_C \setminus \Psi_{sw}\}$	0.1091	0.1106	0.05
101	105	$\{\Psi_C \setminus \Psi_{sw}\}$	0.09211	0.18871	0.05
102	103	$\{\Psi_C \setminus \Psi_{sw}\}$	0.15759	0.15976	0.05
103	104	$\{\Psi_C \setminus \Psi_{sw}\}$	0.33943	0.3441	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
105	106	$\{\Psi_C \setminus \Psi_{SW}\}$	0.1091	0.1106	0.05
105	108	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10886	0.22303	0.05
106	107	$\{\Psi_C \setminus \Psi_{SW}\}$	0.27882	0.28265	0.05
108	109	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2182	0.22121	0.05
108	300	$\{\Psi_C \setminus \Psi_{SW}\}$	0.33496	0.68623	0.05
109	110	$\{\Psi_C \setminus \Psi_{SW}\}$	0.14547	0.14747	0.05
110	111	$\{\Psi_C \setminus \Psi_{SW}\}$	0.27882	0.28265	0.05
110	112	$\{\Psi_C \setminus \Psi_{SW}\}$	0.06061	0.06145	0.05
112	113	$\{\Psi_C \setminus \Psi_{SW}\}$	0.25457	0.25808	0.05
113	114	$\{\Psi_C \setminus \Psi_{SW}\}$	0.15759	0.15976	0.05
135	35	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12561	0.25734	0.05
149	1	$\{\Psi_C \setminus \Psi_{SW}\}$	0.13399	0.27449	0.05
152	52	$\{\Psi_C \setminus \Psi_{SW}\}$	0.13399	0.27449	0.05
160	67	$\{\Psi_C \setminus \Psi_{SW}\}$	0.11724	0.24018	0.05
197	101	$\{\Psi_C \setminus \Psi_{SW}\}$	0.08374	0.17156	0.05
13	152	Ψ_{SW}	0	0.001	0.05
18	135	Ψ_{SW}	0	0.001	0.05
60	160	Ψ_{SW}	0	0.001	0.05
61	610	Ψ_{SW}	0	0.001	0.05
97	197	Ψ_{SW}	0	0.001	0.05
250	251	Ψ_{SW}	0	0.001	0.05
450	451	Ψ_{SW}	0	0.001	0.05
54	94	Ψ_{SW}	0	0.001	0.05
151	300	Ψ_{SW}	0	0.001	0.05
300	350	Ψ_{SW}	0	0.001	0.05
150	149	Ψ_{SW}	0.001	0.008	0.05
83	883	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05
88	888	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
90	890	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05
92	892	$\{\Psi_C \setminus \Psi_{SW}\}$	0	0.001	0.05
33	149	Ψ_{SW}	0	0.001	0.05
25	44	Ψ_{SW}	0	0.001	0.05

7.2 Input data for distribution systems used in section 5.2

7.2.1 Test system S5

Table 7.9. Bus data: test system S5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Cost coefficient for load shedding [\$/MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1	0	0		0.95	1.05	-5	5
1	Ω_{PCTE}		0.1	0.06		0.95	1.05	-5	5
2	$\Omega_{PCTE}, \Omega_{SHED}$		0.09	0.04	1200	0.95	1.05	-5	5
3	$\Omega_{PCTE}, \Omega_{SHED}$		0.12	0.08	1200	0.95	1.05	-5	5
4	Ω_{PCTE}		0.06	0.03		0.95	1.05	-5	5
5	Ω_{PCTE}		0.06	0.02		0.95	1.05	-5	5
6	$\Omega_{PCTE}, \Omega_{SHED}$		0.2	0.1	900	0.95	1.05	-5	5
7	Ω_{PCTE}		0.2	0.1		0.95	1.05	-5	5
8	Ω_{PCTE}		0.06	0.02		0.95	1.05	-5	5
9	$\Omega_{PCTE}, \Omega_{SHED}$		0.06	0.02	1100	0.95	1.05	-5	5
10	Ω_{PCTE}		0.045	0.03		0.95	1.05	-5	5
11	Ω_{PCTE}		0.06	0.035		0.95	1.05	-5	5
12	$\Omega_{PCTE}, \Omega_{SHED}$		0.06	0.035	1100	0.95	1.05	-5	5
13	Ω_{PCTE}		0.12	0.08		0.95	1.05	-5	5
14	Ω_{PCTE}		0.06	0.01		0.95	1.05	-5	5
15	$\Omega_{PCTE}, \Omega_{SHED}$		0.06	0.02	950	0.95	1.05	-5	5
16	Ω_{PCTE}		0.06	0.02		0.95	1.05	-5	5
17	$\Omega_{PCTE}, \Omega_{SHED}$		0.09	0.04	1150	0.95	1.05	-5	5
18	Ω_{PCTE}		0.09	0.04		0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Cost coefficient for load shedding [\$/MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
19	$\Omega_{PCTE}, \Omega_{SHED}$		0.09	0.04	950	0.95	1.05	-5	5
20	Ω_{PCTE}		0.09	0.04		0.95	1.05	-5	5
21	$\Omega_{PCTE}, \Omega_{SHED}$		0.09	0.04	1100	0.95	1.05	-5	5
22	Ω_{PCTE}		0.09	0.05		0.95	1.05	-5	5
23	$\Omega_{PCTE}, \Omega_{SHED}$		0.42	0.2	1100	0.95	1.05	-5	5
24	Ω_{PCTE}		0.42	0.2		0.95	1.05	-5	5
25	Ω_{PCTE}		0.06	0.025		0.95	1.05	-5	5
26	$\Omega_{PCTE}, \Omega_{SHED}$		0.06	0.025	900	0.95	1.05	-5	5
27	$\Omega_{PCTE}, \Omega_{SHED}$		0.06	0.02	1200	0.95	1.05	-5	5
28	Ω_{PCTE}		0.12	0.07		0.95	1.05	-5	5
29	$\Omega_{PCTE}, \Omega_{SHED}$		0.2	0.6	900	0.95	1.05	-5	5
30	$\Omega_{PCTE}, \Omega_{SHED}$		0.15	0.07	950	0.95	1.05	-5	5
31	$\Omega_{PCTE}, \Omega_{SHED}$		0.21	0.1	1050	0.95	1.05	-5	5
32	$\Omega_{PCTE}, \Omega_{SHED}$		0.06	0.04	1100	0.95	1.05	-5	5

Table 7.10. Branch data: test system S5

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1000	1	$\{\Psi_C \setminus \Psi_{SW}\}$	0.05753	0.02932	0.05
1	2	$\{\Psi_C \setminus \Psi_{SW}\}$	0.3076	0.15667	0.05
2	3	$\{\Psi_C \setminus \Psi_{SW}\}$	0.22836	0.1163	0.05
3	4	$\{\Psi_C \setminus \Psi_{SW}\}$	0.23778	0.1211	0.05
4	5	$\{\Psi_C \setminus \Psi_{SW}\}$	0.51099	0.44112	0.05
6	7	Ψ_{SW}	0.44386	0.14668	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
7	8	$\{\Psi_C \setminus \Psi_{SW}\}$	0.64264	0.4617	0.05
8	9	Ψ_{SW}	0.65138	0.4617	0.05
9	10	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12266	0.04056	0.05
10	11	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2336	0.07724	0.05
11	12	$\{\Psi_C \setminus \Psi_{SW}\}$	0.91592	0.72063	0.05
12	13	$\{\Psi_C \setminus \Psi_{SW}\}$	0.33792	0.4448	0.05
13	14	Ψ_{SW}	0.36874	0.32818	0.05
14	15	$\{\Psi_C \setminus \Psi_{SW}\}$	0.46564	0.34004	0.05
15	16	$\{\Psi_C \setminus \Psi_{SW}\}$	0.80424	1.07378	0.05
16	17	Ψ_{SW}	0.45671	0.35813	0.05
1	18	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10232	0.09764	0.05
18	19	$\{\Psi_C \setminus \Psi_{SW}\}$	0.93851	0.84567	0.05
19	20	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2555	0.29849	0.05
20	21	$\{\Psi_C \setminus \Psi_{SW}\}$	0.4423	0.58481	0.05
2	22	$\{\Psi_C \setminus \Psi_{SW}\}$	0.28152	0.19236	0.05
22	23	$\{\Psi_C \setminus \Psi_{SW}\}$	0.56028	0.44243	0.05
23	24	$\{\Psi_C \setminus \Psi_{SW}\}$	0.55904	0.43743	0.05
25	26	$\{\Psi_C \setminus \Psi_{SW}\}$	0.17732	0.09028	0.05
26	27	$\{\Psi_C \setminus \Psi_{SW}\}$	0.66074	0.58256	0.05
27	28	Ψ_{SW}	0.50176	0.43712	0.05
28	29	$\{\Psi_C \setminus \Psi_{SW}\}$	0.31664	0.16128	0.05
29	30	$\{\Psi_C \setminus \Psi_{SW}\}$	0.60795	0.60084	0.05
30	31	Ψ_{SW}	0.19373	0.2258	0.05
31	32	Ψ_{SW}	0.21276	0.33081	0.05
7	20	Ψ_{SW}	1.24785	1.24785	0.05
8	14	Ψ_{SW}	1.24785	1.24785	0.05
11	21	Ψ_{SW}	1.24785	1.24785	0.05
17	32	Ψ_{SW}	0.31196	0.31196	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
24	28	Ψ_{sw}	0.31196	0.31196	0.05

7.2.2 Test system S6

Table 7.11. Bus data: test system S6

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	(Fixed) active power generation [MW]	Lower bound for reactive power generation [MVar]	Upper bound for reactive power generation [MVar]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1.05	0	0				0.95	1.05	-5	5
1	Ω_{PCTE}		0.03	0.018				0.95	1.05	-5	5
2	$\Omega_{PCTE}, \Omega_{CURT}$		0.027	0.012	0.09	-0.018	-0.018	0.95	1.05	-5	5
3	$\Omega_{PCTE}, \Omega_{CURT}$		0.036	0.024	0.089	-0.018	-0.018	0.95	1.05	-5	5
4	Ω_{PCTE}		0.018	0.009				0.95	1.05	-5	5
5	Ω_{PCTE}		0.018	0.006				0.95	1.05	-5	5
6	$\Omega_{PCTE}, \Omega_{CURT}$		0.06	0.03	0.058	-0.012	-0.012	0.95	1.05	-5	5
7	$\Omega_{PCTE}, \Omega_{CURT}$		0.06	0.03	0.062	-0.013	-0.013	0.95	1.05	-5	5
8	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.006	0.138	-0.028	-0.028	0.95	1.05	-5	5
9	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.006	0.068	-0.014	-0.014	0.95	1.05	-5	5
10	Ω_{PCTE}		0.0135	0.009				0.95	1.05	-5	5
11	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.0105	0.088	-0.018	-0.018	0.95	1.05	-5	5
12	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.0105	0.065	-0.013	-0.013	0.95	1.05	-5	5
13	Ω_{PCTE}		0.036	0.024				0.95	1.05	-5	5
14	Ω_{PCTE}		0.018	0.003				0.95	1.05	-5	5
15	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.006	0.075	-0.015	-0.015	0.95	1.05	-5	5
16	Ω_{PCTE}		0.018	0.006				0.95	1.05	-5	5
17	Ω_{PCTE}		0.027	0.012				0.95	1.05	-5	5
18	$\Omega_{PCTE}, \Omega_{CURT}$		0.027	0.012	0.1	-0.02	-0.02	0.95	1.05	-5	5
19	$\Omega_{PCTE}, \Omega_{CURT}$		0.027	0.012	0.069	-0.014	-0.014	0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	(Fixed) active power generation [MW]	Lower bound for reactive power generation [MVar]	Upper bound for reactive power generation [MVar]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
20	Ω_{PCTE}		0.027	0.012				0.95	1.05	-5	5
21	$\Omega_{PCTE}, \Omega_{CURT}$		0.027	0.012	0.168	-0.034	-0.034	0.95	1.05	-5	5
22	$\Omega_{PCTE}, \Omega_{CURT}$		0.027	0.015	0.146	-0.03	-0.03	0.95	1.05	-5	5
23	$\Omega_{PCTE}, \Omega_{CURT}$		0.126	0.06	0.123	-0.025	-0.025	0.95	1.05	-5	5
24	Ω_{PCTE}		0.126	0.06				0.95	1.05	-5	5
25	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.0075	0.153	-0.031	-0.031	0.95	1.05	-5	5
26	Ω_{PCTE}		0.018	0.0075				0.95	1.05	-5	5
27	Ω_{PCTE}		0.018	0.006				0.95	1.05	-5	5
28	$\Omega_{PCTE}, \Omega_{CURT}$		0.036	0.021	0.126	-0.026	-0.026	0.95	1.05	-5	5
29	$\Omega_{PCTE}, \Omega_{CURT}$		0.06	0.18	0.085	-0.017	-0.017	0.95	1.05	-5	5
30	$\Omega_{PCTE}, \Omega_{CURT}$		0.045	0.021	0.094	-0.019	-0.019	0.95	1.05	-5	5
31	$\Omega_{PCTE}, \Omega_{CURT}$		0.063	0.03	0.089	-0.018	-0.018	0.95	1.05	-5	5
32	$\Omega_{PCTE}, \Omega_{CURT}$		0.018	0.012	0.114	-0.023	-0.023	0.95	1.05	-5	5

Table 7.12. Branch data: test system S6

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1000	1	$\{\Psi_C \setminus \Psi_{SW}\}$	0.05753	0.02932	0.05
1	2	$\{\Psi_C \setminus \Psi_{SW}\}$	0.3076	0.15667	0.05
2	3	$\{\Psi_C \setminus \Psi_{SW}\}$	0.22836	0.1163	0.05
3	4	$\{\Psi_C \setminus \Psi_{SW}\}$	0.23778	0.1211	0.05
4	5	$\{\Psi_C \setminus \Psi_{SW}\}$	0.51099	0.44112	0.05
5	6	Ψ_{SW}	0.1168	0.38608	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
6	7	$\{\Psi_C \setminus \Psi_{SW}\}$	0.44386	0.14668	0.05
7	8	Ψ_{SW}	0.64264	0.4617	0.05
8	9	$\{\Psi_C \setminus \Psi_{SW}\}$	0.65138	0.4617	0.05
9	10	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12266	0.04056	0.05
10	11	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2336	0.07724	0.05
11	12	$\{\Psi_C \setminus \Psi_{SW}\}$	0.91592	0.72063	0.05
12	13	Ψ_{SW}	0.33792	0.4448	0.05
13	14	$\{\Psi_C \setminus \Psi_{SW}\}$	0.36874	0.32818	0.05
14	15	$\{\Psi_C \setminus \Psi_{SW}\}$	0.46564	0.34004	0.05
15	16	$\{\Psi_C \setminus \Psi_{SW}\}$	0.80424	1.07378	0.05
16	17	$\{\Psi_C \setminus \Psi_{SW}\}$	0.45671	0.35813	0.05
1	18	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10232	0.09764	0.05
18	19	$\{\Psi_C \setminus \Psi_{SW}\}$	0.93851	0.84567	0.05
19	20	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2555	0.29849	0.05
20	21	$\{\Psi_C \setminus \Psi_{SW}\}$	0.4423	0.58481	0.05
2	22	$\{\Psi_C \setminus \Psi_{SW}\}$	0.28152	0.19236	0.05
22	23	$\{\Psi_C \setminus \Psi_{SW}\}$	0.56028	0.44243	0.05
23	24	$\{\Psi_C \setminus \Psi_{SW}\}$	0.55904	0.43743	0.05
5	25	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12666	0.06451	0.05
25	26	Ψ_{SW}	0.17732	0.09028	0.05
26	27	$\{\Psi_C \setminus \Psi_{SW}\}$	0.66074	0.58256	0.05
27	28	$\{\Psi_C \setminus \Psi_{SW}\}$	0.50176	0.43712	0.05
28	29	$\{\Psi_C \setminus \Psi_{SW}\}$	0.31664	0.16128	0.05
29	30	Ψ_{SW}	0.60795	0.60084	0.05
30	31	Ψ_{SW}	0.19373	0.2258	0.05
31	32	Ψ_{SW}	0.21276	0.33081	0.05
7	20	Ψ_{SW}	1.24785	1.24785	0.05
8	14	Ψ_{SW}	1.24785	1.24785	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
11	21	Ψ_{SW}	1.24785	1.24785	0.05
17	32	Ψ_{SW}	0.31196	0.31196	0.05
24	28	Ψ_{SW}	0.31196	0.31196	0.05

7.2.3 Test system S7

Table 7.13. Bus data: test system S7

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for active power generation [MW]	Upper bound for active power generation [MW]	Lower bound for reactive power generation [MVar]	Upper bound for reactive power generation [MVar]	Variable generation costs [\$/MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1.025	0	0						0.95	1.05	-5	5
1	Ω_{PCTE}		0.11	0.066						0.95	1.05	-5	5
2	Ω_{PCTE}		0.099	0.044						0.95	1.05	-5	5
3	Ω_{PCTE}		0.132	0.088						0.95	1.05	-5	5
4	Ω_{PCTE}		0.066	0.033						0.95	1.05	-5	5
5	Ω_{PCTE}		0.066	0.022						0.95	1.05	-5	5
6	Ω_{PCTE}		0.22	0.11						0.95	1.05	-5	5
7	$\Omega_{PCTE}, \Omega_{CTRPO}$		0.22	0.11	0	0.5	0	0.1643	110	0.95	1.05	-5	5
8	Ω_{PCTE}		0.066	0.022						0.95	1.05	-5	5
9	Ω_{PCTE}		0.066	0.022						0.95	1.05	-5	5
10	Ω_{PCTE}		0.0495	0.033						0.95	1.05	-5	5
11	Ω_{PCTE}		0.066	0.0385						0.95	1.05	-5	5
12	Ω_{PCTE}		0.066	0.0385						0.95	1.05	-5	5
13	Ω_{PCTE}		0.132	0.088						0.95	1.05	-5	5
14	$\Omega_{PCTE}, \Omega_{CTRPO}$		0.066	0.011	0	0.3	0	0.0986	118	0.95	1.05	-5	5
15	Ω_{PCTE}		0.066	0.022						0.95	1.05	-5	5
16	Ω_{PCTE}		0.066	0.022						0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for active power generation [MW]	Upper bound for active power generation [MW]	Lower bound for reactive power generation [MVar]	Upper bound for reactive power generation [MVar]	Variable generation costs [\$/MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
17	$\Omega_{PCTE}, \Omega_{CTRPO}$		0.099	0.044	0	0.3	0	0.0986	118	0.95	1.05	-5	5
18	Ω_{PCTE}		0.099	0.044						0.95	1.05	-5	5
19	Ω_{PCTE}		0.099	0.044						0.95	1.05	-5	5
20	Ω_{PCTE}		0.099	0.044						0.95	1.05	-5	5
21	Ω_{PCTE}		0.099	0.044						0.95	1.05	-5	5
22	Ω_{PCTE}		0.099	0.055						0.95	1.05	-5	5
23	Ω_{PCTE}		0.462	0.22						0.95	1.05	-5	5
24	Ω_{PCTE}		0.462	0.22						0.95	1.05	-5	5
25	Ω_{PCTE}		0.066	0.0275						0.95	1.05	-5	5
26	Ω_{PCTE}		0.066	0.0275						0.95	1.05	-5	5
27	Ω_{PCTE}		0.066	0.022						0.95	1.05	-5	5
28	Ω_{PCTE}		0.132	0.077						0.95	1.05	-5	5
29	$\Omega_{PCTE}, \Omega_{CTRPO}$		0.22	0.66	0	0.3	0	0.0986	145	0.95	1.05	-5	5
30	Ω_{PCTE}		0.165	0.077						0.95	1.05	-5	5
31	Ω_{PCTE}		0.231	0.11						0.95	1.05	-5	5
32	Ω_{PCTE}		0.066	0.044						0.95	1.05	-5	5

Table 7.14. Branch data: test system S7

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1000	1	Ψ_{sw}	0.05753	0.02932	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1	2	$\{\Psi_C \setminus \Psi_{SW}\}$	0.3076	0.15667	0.05
2	3	$\{\Psi_C \setminus \Psi_{SW}\}$	0.22836	0.1163	0.05
3	4	$\{\Psi_C \setminus \Psi_{SW}\}$	0.23778	0.1211	0.05
4	5	$\{\Psi_C \setminus \Psi_{SW}\}$	0.51099	0.44112	0.05
5	6	Ψ_{SW}	0.1168	0.38608	0.05
6	7	$\{\Psi_C \setminus \Psi_{SW}\}$	0.44386	0.14668	0.05
7	8	Ψ_{SW}	0.64264	0.4617	0.05
8	9	$\{\Psi_C \setminus \Psi_{SW}\}$	0.65138	0.4617	0.05
9	10	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12266	0.04056	0.05
10	11	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2336	0.07724	0.05
11	12	$\{\Psi_C \setminus \Psi_{SW}\}$	0.91592	0.72063	0.05
12	13	Ψ_{SW}	0.33792	0.4448	0.05
13	14	$\{\Psi_C \setminus \Psi_{SW}\}$	0.36874	0.32818	0.05
14	15	$\{\Psi_C \setminus \Psi_{SW}\}$	0.46564	0.34004	0.05
15	16	$\{\Psi_C \setminus \Psi_{SW}\}$	0.80424	1.07378	0.05
16	17	$\{\Psi_C \setminus \Psi_{SW}\}$	0.45671	0.35813	0.05
1	18	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10232	0.09764	0.05
18	19	$\{\Psi_C \setminus \Psi_{SW}\}$	0.93851	0.84567	0.05
19	20	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2555	0.29849	0.05
20	21	$\{\Psi_C \setminus \Psi_{SW}\}$	0.4423	0.58481	0.05
2	22	$\{\Psi_C \setminus \Psi_{SW}\}$	0.28152	0.19236	0.05
22	23	$\{\Psi_C \setminus \Psi_{SW}\}$	0.56028	0.44243	0.05
23	24	$\{\Psi_C \setminus \Psi_{SW}\}$	0.55904	0.43743	0.05
5	25	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12666	0.06451	0.05
25	26	Ψ_{SW}	0.17732	0.09028	0.05
26	27	$\{\Psi_C \setminus \Psi_{SW}\}$	0.66074	0.58256	0.05
27	28	$\{\Psi_C \setminus \Psi_{SW}\}$	0.50176	0.43712	0.05
28	29	$\{\Psi_C \setminus \Psi_{SW}\}$	0.31664	0.16128	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
29	30	Ψ_{SW}	0.60795	0.60084	0.05
30	31	Ψ_{SW}	0.19373	0.2258	0.05
31	32	Ψ_{SW}	0.21276	0.33081	0.05
7	20	Ψ_{SW}	1.24785	1.24785	0.05
8	14	Ψ_{SW}	1.24785	1.24785	0.05
11	21	Ψ_{SW}	1.24785	1.24785	0.05
17	32	$\{\Psi_C \setminus \Psi_{SW}\}$	0.31196	0.31196	0.05
24	28	$\{\Psi_C \setminus \Psi_{SW}\}$	0.31196	0.31196	0.05

7.2.4 Test system S8

Table 7.15. Bus data: test system S8

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Annualized cost of candidate capacitor [\$]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1				0.95	1.05	-5	5
1	Ω_{PCTE}		0.125	0.075		0.95	1.05	-5	5
2	Ω_{PCTE}		0.1125	0.05		0.95	1.05	-5	5
3	Ω_{PCTE}		0.15	0.1		0.95	1.05	-5	5
4	Ω_{PCTE}		0.075	0.0375		0.95	1.05	-5	5
5	Ω_{PCTE}		0.075	0.025		0.95	1.05	-5	5
6	Ω_{PCTE}		0.25	0.125		0.95	1.05	-5	5
7	Ω_{PCTE}		0.25	0.125		0.95	1.05	-5	5
8	Ω_{PCTE}		0.075	0.025		0.95	1.05	-5	5
9	Ω_{PCTE}		0.075	0.025		0.95	1.05	-5	5
10	Ω_{PCTE}		0.05625	0.0375		0.95	1.05	-5	5
11	Ω_{PCTE}		0.075	0.04375		0.95	1.05	-5	5
12	Ω_{PCTE}		0.075	0.04375		0.95	1.05	-5	5
13	Ω_{PCTE}		0.15	0.1		0.95	1.05	-5	5
14	Ω_{PCTE}		0.075	0.0125		0.95	1.05	-5	5
15	Ω_{PCTE}		0.075	0.025		0.95	1.05	-5	5
16	Ω_{PCTE}		0.075	0.025		0.95	1.05	-5	5
17	Ω_{PCTE}		0.1125	0.05		0.95	1.05	-5	5
18	Ω_{PCTE}		0.1125	0.05		0.95	1.05	-5	5
19	Ω_{PCTE}		0.1125	0.05		0.95	1.05	-5	5
20	Ω_{PCTE}		0.1125	0.05		0.95	1.05	-5	5
21	Ω_{PCTE}		0.1125	0.05		0.95	1.05	-5	5

Bus #	Set(s) to which bus pertain	Reference voltage magnitude [p.u.]	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Annualized cost of candidate capacitor [\$]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
22	Ω_{PCTE}		0.1125	0.0625		0.95	1.05	-5	5
23	Ω_{PCTE}		0.525	0.25		0.95	1.05	-5	5
24	Ω_{PCTE}		0.525	0.25		0.95	1.05	-5	5
25	Ω_{PCTE}		0.075	0.03125		0.95	1.05	-5	5
26	Ω_{PCTE}		0.075	0.03125		0.95	1.05	-5	5
27	Ω_{PCTE}		0.075	0.025		0.95	1.05	-5	5
28	Ω_{PCTE}		0.15	0.0875		0.95	1.05	-5	5
29	Ω_{PCTE}		0.25	0.75		0.95	1.05	-5	5
30	Ω_{PCTE}		0.1875	0.0875		0.95	1.05	-5	5
31	Ω_{PCTE}		0.2625	0.125		0.95	1.05	-5	5
32	Ω_{PCTE}		0.075	0.05		0.95	1.05	-5	5
2000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$	1				0.95	1.05	-5	5
200	Ω_{PCTE}					0.95	1.05	-5	5
805	$\Omega_{ZCTE}, \Omega_{CAP}$		0	-0.2	9000	0.95	1.05	-5	5
811	$\Omega_{ZCTE}, \Omega_{CAP}$		0	-0.2	9000	0.95	1.05	-5	5
831	$\Omega_{ZCTE}, \Omega_{CAP}$		0	-0.2	9000	0.95	1.05	-5	5

Table 7.16. Branch data: test system S8

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]	Annualized cost of candidate facility [\$]
1000	1	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.05753	0.02932	0.05	
1	2	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.3076	0.15667	0.04	
2	3	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.22836	0.1163	0.04	
3	4	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.23778	0.1211	0.04	

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]	Annualized cost of candidate facility [\$]
4	5	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.51099	0.44112	0.035	
5	6	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.1168	0.38608	0.035	
6	7	Ψ_{SW}	0.44386	0.14668	0.035	
7	8	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.64264	0.4617	0.035	
8	9	Ψ_{SW}	0.65138	0.4617	0.035	
9	10	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.12266	0.04056	0.03	
10	11	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.2336	0.07724	0.03	
11	12	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.91592	0.72063	0.03	
12	13	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.33792	0.4448	0.03	
13	14	Ψ_{SW}	0.36874	0.32818	0.03	
14	15	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.46564	0.34004	0.03	
15	16	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.80424	1.07378	0.03	
16	17	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.45671	0.35813	0.03	
1	18	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.10232	0.09764	0.04	
18	19	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.93851	0.84567	0.04	
19	20	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.2555	0.29849	0.04	
20	21	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.4423	0.58481	0.035	
2	22	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.28152	0.19236	0.04	
22	23	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.56028	0.44243	0.04	
23	24	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.55904	0.43743	0.035	
5	25	Ψ_{SW}	0.12666	0.06451	0.035	
25	26	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.17732	0.09028	0.035	
26	27	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.66074	0.58256	0.035	
27	28	Ψ_{SW}	0.50176	0.43712	0.035	
28	29	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.31664	0.16128	0.035	
29	30	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.60795	0.60084	0.035	
30	31	$\{\Psi_C \setminus \{\Psi_{SW} \cup \Psi_{CD}\}\}$	0.19373	0.2258	0.03	

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]	Annualized cost of candidate facility [\$]
31	32	Ψ_{SW}	0.21276	0.33081	0.03	
7	20	Ψ_{SW}	1.24785	1.24785	0.035	
8	14	Ψ_{SW}	1.24785	1.24785	0.035	
11	21	Ψ_{SW}	1.24785	1.24785	0.03	
17	32	Ψ_{SW}	0.31196	0.31196	0.03	
24	28	Ψ_{SW}	0.31196	0.31196	0.035	
2000	200	Ψ_{CD}	0.05292	0.03226	0.05	48000
200	13	Ψ_{CD}	0.36593	0.30952	0.04	7800
200	15	Ψ_{CD}	0.33778	0.28571	0.04	7200
13	8	Ψ_{CD}	0.42223	0.35713	0.04	9000
9	21	Ψ_{CD}	0.54748	0.4742	0.035	8400
15	26	Ψ_{CD}	0.56297	0.47618	0.04	12000
15	30	Ψ_{CD}	0.60125	0.52856	0.05	11000
5	805	Ψ_{SW}	0	0.001	0.05	
11	811	Ψ_{SW}	0	0.001	0.05	
31	831	Ψ_{SW}	0	0.001	0.05	

7.2.5 Test system S9

Table 7.17. Bus data: test system S9

Bus #	Set(s) to which bus pertain	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
1000	$\Omega_{SLACK}, \Omega_{REF}, \Omega_{ROOT}$			0.8	1.05	-5	5
1	Ω_{ZCTE}	0.1	0.06	0.8	1.05	-5	5
2	Ω_{ICTE}	0.09	0.04	0.8	1.05	-5	5
3	Ω_{ZCTE}	0.12	0.08	0.8	1.05	-5	5
4	Ω_{ZCTE}	0.06	0.03	0.8	1.05	-5	5
5	Ω_{PCTE}	0.06	0.02	0.8	1.05	-5	5
6	Ω_{PCTE}	0.2	0.1	0.8	1.05	-5	5
7	Ω_{ZCTE}	0.2	0.1	0.8	1.05	-5	5
8	Ω_{ZCTE}	0.06	0.02	0.8	1.05	-5	5
9	Ω_{PCTE}	0.06	0.02	0.8	1.05	-5	5
10	Ω_{ZCTE}	0.045	0.03	0.8	1.05	-5	5
11	Ω_{ICTE}	0.06	0.035	0.8	1.05	-5	5
12	Ω_{PCTE}	0.06	0.035	0.8	1.05	-5	5
13	Ω_{ZCTE}	0.12	0.08	0.8	1.05	-5	5
14	Ω_{PCTE}	0.06	0.01	0.8	1.05	-5	5
15	Ω_{PCTE}	0.06	0.02	0.8	1.05	-5	5
16	Ω_{PCTE}	0.06	0.02	0.8	1.05	-5	5
17	Ω_{ICTE}	0.09	0.04	0.8	1.05	-5	5
18	Ω_{ZCTE}	0.09	0.04	0.8	1.05	-5	5
19	Ω_{PCTE}	0.09	0.04	0.8	1.05	-5	5

Bus #	Set(s) to which bus pertain	Nominal value of active load [MW]	Nominal value of reactive load [MW]	Lower bound for voltage magnitude [p.u.]	Upper bound for voltage magnitude [p.u.]	Lower bound for voltage angle [°]	Upper bound for voltage angle [°]
20	Ω_{ZCTE}	0.09	0.04	0.8	1.05	-5	5
21	Ω_{ZCTE}	0.09	0.04	0.8	1.05	-5	5
22	Ω_{ZCTE}	0.09	0.05	0.8	1.05	-5	5
23	Ω_{PCTE}	0.42	0.2	0.8	1.05	-5	5
24	Ω_{ZCTE}	0.42	0.2	0.8	1.05	-5	5
25	Ω_{PCTE}	0.06	0.025	0.8	1.05	-5	5
26	Ω_{ICTE}	0.06	0.025	0.8	1.05	-5	5
27	Ω_{ICTE}	0.06	0.02	0.8	1.05	-5	5
28	Ω_{ZCTE}	0.12	0.07	0.8	1.05	-5	5
29	Ω_{ICTE}	0.2	0.6	0.8	1.05	-5	5
30	Ω_{ICTE}	0.15	0.07	0.8	1.05	-5	5
31	Ω_{ZCTE}	0.21	0.1	0.8	1.05	-5	5
32	Ω_{PCTE}	0.06	0.04	0.8	1.05	-5	5

Table 7.18. Branch data: test system S9

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
1000	1	$\{\Psi_C \setminus \Psi_{SW}\}$	0.05753	0.02932	0.05
1	2	$\{\Psi_C \setminus \Psi_{SW}\}$	0.3076	0.15667	0.05
2	3	$\{\Psi_C \setminus \Psi_{SW}\}$	0.22836	0.1163	0.05
3	4	$\{\Psi_C \setminus \Psi_{SW}\}$	0.23778	0.1211	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
4	5	$\{\Psi_C \setminus \Psi_{SW}\}$	0.51099	0.44112	0.05
5	6	$\{\Psi_C \setminus \Psi_{SW}\}$	0.1168	0.38608	0.05
6	7	Ψ_{SW}	0.44386	0.14668	0.05
7	8	$\{\Psi_C \setminus \Psi_{SW}\}$	0.64264	0.4617	0.05
8	9	Ψ_{SW}	0.65138	0.4617	0.05
9	10	$\{\Psi_C \setminus \Psi_{SW}\}$	0.12266	0.04056	0.05
10	11	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2336	0.07724	0.05
11	12	$\{\Psi_C \setminus \Psi_{SW}\}$	0.91592	0.72063	0.05
12	13	$\{\Psi_C \setminus \Psi_{SW}\}$	0.33792	0.4448	0.05
13	14	Ψ_{SW}	0.36874	0.32818	0.05
14	15	$\{\Psi_C \setminus \Psi_{SW}\}$	0.46564	0.34004	0.05
15	16	$\{\Psi_C \setminus \Psi_{SW}\}$	0.80424	1.07378	0.05
16	17	$\{\Psi_C \setminus \Psi_{SW}\}$	0.45671	0.35813	0.05
1	18	$\{\Psi_C \setminus \Psi_{SW}\}$	0.10232	0.09764	0.05
18	19	$\{\Psi_C \setminus \Psi_{SW}\}$	0.93851	0.84567	0.05
19	20	$\{\Psi_C \setminus \Psi_{SW}\}$	0.2555	0.29849	0.05
20	21	$\{\Psi_C \setminus \Psi_{SW}\}$	0.4423	0.58481	0.05
2	22	$\{\Psi_C \setminus \Psi_{SW}\}$	0.28152	0.19236	0.05
22	23	$\{\Psi_C \setminus \Psi_{SW}\}$	0.56028	0.44243	0.05
23	24	$\{\Psi_C \setminus \Psi_{SW}\}$	0.55904	0.43743	0.05
5	25	Ψ_{SW}	0.12666	0.06451	0.05
25	26	$\{\Psi_C \setminus \Psi_{SW}\}$	0.17732	0.09028	0.05
26	27	$\{\Psi_C \setminus \Psi_{SW}\}$	0.66074	0.58256	0.05
27	28	Ψ_{SW}	0.50176	0.43712	0.05
28	29	$\{\Psi_C \setminus \Psi_{SW}\}$	0.31664	0.16128	0.05

FROM bus	TO bus	Set(s) to which branch pertain	Branch resistance [p.u.]	Branch reactance [p.u.]	Maximum admissible current [p.u.]
29	30	$\{\Psi_C \setminus \Psi_{SW}\}$	0.60795	0.60084	0.05
30	31	$\{\Psi_C \setminus \Psi_{SW}\}$	0.19373	0.2258	0.05
31	32	Ψ_{SW}	0.21276	0.33081	0.05
7	20	Ψ_{SW}	1.24785	1.24785	0.05
8	14	Ψ_{SW}	1.24785	1.24785	0.05
11	21	Ψ_{SW}	1.24785	1.24785	0.05
17	32	Ψ_{SW}	0.31196	0.31196	0.05
24	28	Ψ_{SW}	0.31196	0.31196	0.05

8 APPENDIX B: AN ALTERNATIVE MILP REFORMULATION OF THE ACOPF IN DISTRIBUTION SYSTEMS

In the course of the research activities that led to the present dissertation, an alternative MILP reformulation of the ACOPF in distribution systems has been investigated. This alternative formulation has been abandoned at early stages of the research activities due to its performance being inferior, with respect to accuracy and computational requirements, to the formulation presented in sections 4.1 to 4.3 of this document. For the sake of didactics, the alternative formulation is thoroughly presented below. The nomenclature used for the presentation of the alternative formulation is consistent with that used in chapter 4, except when otherwise noted.

8.1 Main differences with respect to the formulation presented in chapter 4

The alternative MILP reformulation of the ACOPF in distribution systems is similar to that presented in chapter 4 in various aspects, but differs from it mainly with respect to the construction of piecewise-linear approximations of non-linear, non-convex functions. Each and every segment of the piecewise-linear approximations of non-linear functions in the formulation presented in chapter 4 is obtained by affine combinations of its vertices. In the alternative formulation, each segment of the piecewise-linear approximation consists of a *constant value*, which is deemed as representative of the values that the non-linear function assumes between the vertices of a partition of its domain. Figure 8.1 provides the reader with insight about the differences among the piecewise-linearization with affine combinations of the vertices and the piecewise-linearization considering constant values of the non-linear function. From this figure, it is clear that the piecewise piecewise-linearization obtained by considering constant values of the function within a partition of its domain has the approximate shape of a *staircase*. For that reason, we will refer to this as a *staircase-shaped piecewise-linear approximation* – or simply *SSPL approximation*.

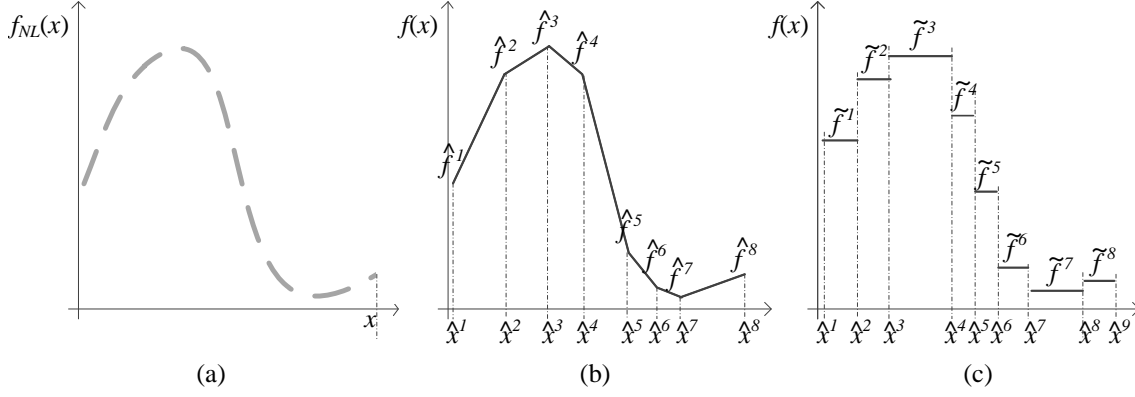


Figure 8.1: Non-linear function $f_{NL}(x)$ (a); piecewise-linearization $f(x)$ via affine combination of vertices (b); piecewise-linearization $f(x)$ considering constant values within the vertices of a partition of the domain (SSPL approximation) (c).

Figure 8.1 also points to a difference in the nomenclature used for defining the piecewise-linear approximations. In previous chapters of this dissertation we referred to the values $\hat{f}^i - \hat{f}^1$ to \hat{f}^8 in part (b) of Figure 8.1 – as *evaluated values* (the vertices of the linear segments). The SSPL approximation defined in this chapter no longer makes use of affine combinations of *evaluated values*, but rather employs *constant values* through which the function $f_{NL}(x)$ is represented within the partition of the domain. These constant values will be referred to as *representative values* in this chapter, and will be denoted by \tilde{f}^j . As indicated in Figure 8.1, the representative value \tilde{f}^j is a single value chosen within the interval $f_{NL}(\hat{x}^j) \leq \tilde{f}^j \leq f_{NL}(\hat{x}^{j+1})$, where $f_{NL}(x)$ is the non-linear function to be approximated.

This difference in the approximation of the non-linear functions requires the rewriting of several constraints of the ACOPF problem presented in chapter 4 – notably, those that relate power injections at buses with the correspondent current injections. By inspection of the constraints presented in this chapter 8, the reader will notice that the approximation of non-linear functions by *constant values* allows that the very nature of the functions being approximated changes: some of the functions for which piecewise-linear approximation were used in chapter 4 (such as ξ_k and ζ_k) need no longer to be approximated, as the (linear) constraints in which these functions were used are rewritten with a different arrangement of the decision variables. Particularly, as shown in section 8.2, the alternative MILP reformulation requires only the approximation of non-linear functions of a single decision variable in order to obtain the current injections demanded by *constant-power loads* and *generators*.

8.2 Mathematical formulation

Analogously to section 4.2, this section begins with the presentation of the constraints employed for modeling the behavior of the network and enforcing operating limits (subsection 8.2.1). For the constraints that do not demand any modification with respect to the formulation presented in chapter 4, we will simply make direct reference to the associated equations of section 4.2.1.

Objective functions for selected distribution system operations and expansion planning applications will be dealt with in subsection 8.2.2.

8.2.1 Constraints: modeling electrical behavior and enforcing operating limits

8.2.1.1 *Kirchhoff's Laws*

The alternative formulation requires no modifications to the constraints presented in subsection 4.2.1.1. Thus, those constraints may be promptly incorporated to the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.1.2 *Operating limits*

8.2.1.2.1 Bounds on bus voltage magnitudes

The magnitude of the voltage at bus k , V_k , is a non-linear, non-convex function of the real and imaginary components of the voltage at this bus. Analogously, the squared value of the bus voltage magnitude, V_k^2 , is also a non-linear, non-convex function of the associated real and imaginary components. As V_k may only assume non-negative values, and as the square function is strictly monotonically increasing in the non-negative domain, bounding V_k^2 within the interval $(\underline{V}_k)^2 \leq V_k^2 \leq (\overline{V}_k)^2$ equals bounding V_k within the interval $\underline{V}_k \leq V_k \leq \overline{V}_k$. This fact will be explored in the alternative MILP reformulation of the ACOPF for distribution systems – as the term V_k^2 will be used for the formulation of other constraints of the alternative MILP

formulation, a choice is made to use a constraint analogous to $(\underline{V}_k)^2 \leq V_k \leq (\overline{V}_k)^2$ to bound the bus voltage magnitudes.

In order to do that, the non-linear term V_k^2 will be substituted by an auxiliary continuous decision variable, μ_k . It is thus needed to approximate the following non-linear function:

$$\mu_k = (V_k^{re})^2 + (V_k^{im})^2, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (357)$$

It is clear that the non-linear function presented above is separable, and may be rewritten as:

$$\mu_k = \mu_k^{re} + \mu_k^{im}, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (358)$$

$$\mu_k^{re} = (V_k^{re})^2, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (359)$$

$$\mu_k^{im} = (V_k^{im})^2, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (360)$$

where:

$\mu_k^{re}; \mu_k^{im}$ Auxiliary, continuous decision variables.

Thus, instead of approximating the non-linear function of two decision variables expressed by equation (357), it is only required to separately approximate each of the functions of a single variable expressed by equations (359) and (360), and to summate them to obtain μ_k , as indicated in equation (358).

The approximation of the function $\mu_k^{re} = (V_k^{re})^2$ will be dealt with first. A SSPL approximation will be used for the reformulation of this function. The first step for using a SSPL approximation is to discover *which partition of the domain* (which partition comprised within two consecutive vertices that correspond to *evaluation points*) corresponds to the value of the decision variable V_k^{re} (the argument of the non-linear function) at a given solution. In order to do that, the following constraints are employed:

$$\sum_{r \in \Gamma^{re}} \lambda_k^{re,r} \cdot \hat{V}_k^{re,r} = V_k^{re}, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (361)$$

$$\sum_{r \in \Gamma^{re}} \lambda_k^{re,r} = 1, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (362)$$

$$\sum_{r \in \Gamma^{re}} x_k^r = 1, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (363)$$

$$\lambda_k^{re,1} \leq x_k^1, \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (364)$$

$$\lambda_k^{re,r} \leq x_k^{r-1} + x_k^r, \forall r \in \{\Gamma^{re} \setminus \{1\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (365)$$

where:

- Γ^{re} Set of indices for *evaluation points* $\hat{V}_k^{re,r}$ and associated variables;
- $\hat{V}_k^{re,r}$ Evaluation points of real component of voltage at bus k ;
- $\lambda_k^{re,r}$ Weights used for expressing the value of the *argument* V_k^{re} as an affine combination of the *evaluation points*;
- x_k^r Auxiliary binary decision variable.

The reader will notice that the previous equations are very similar to those used in chapter 4 to ensure that the set of weights $\lambda_k^{r,s}$ corresponds to a SOS2. Whenever these equations are enforced, it is possible to use the information of the auxiliary variables x_k^r to check in which partition of the domain the variable V_k^{re} is:

- If $x_k^r = 1$, with $r \in \{\Gamma^{re} \setminus \{|\Gamma^{re}|\}\}$, V_k^{re} is within the partition defined by the interval $\hat{V}_k^{re,r} \leq V_k^{re} \leq \hat{V}_k^{re,r+1}$;
- If $x_k^{|\Gamma^{re}|} = 1$, then $V_k^{re} = \hat{V}_k^{re,|\Gamma^{re}|}$.

By using the information of the partition of the domain within which the argument V_k^{re} is located, it is possible to employ the following disjunctive constraints to construct a SSPL approximation of the function $\mu_k^{re} = (V_k^{re})^2$:

$$M_k^{r,re,l} \cdot (1 - x_k^r) \leq \mu_k^{re} - \tilde{\mu}_k^{re,r} \leq M_k^{r,re,u} \cdot (1 - x_k^r), \forall r \in \{\Gamma^{re} \setminus \{(|\Gamma^{re}| - 1), |\Gamma^{re}|\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (366)$$

$$M_k^{(|\Gamma^{re}|-1),re,l} \cdot (1 - x_k^{(|\Gamma^{re}|-1)} - x_k^{|\Gamma^{re}|}) \leq \mu_k^{re} - \tilde{\mu}_k^{re,(|\Gamma^{re}|-1)} \leq M_k^{(|\Gamma^{re}|-1),re,u} \cdot (1 - x_k^{(|\Gamma^{re}|-1)} - x_k^{|\Gamma^{re}|}), \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (367)$$

where:

$$M_k^{r,re,l}; M_k^{r,re,u}$$

Disjunctive constants. These parameters need to be defined only for $r \in \{\Gamma^{re} \setminus \{|\Gamma^{re}|\}\}$;

$\tilde{\mu}_k^{re,r}$ *Representative values* of the function $\mu_k^{re} = (V_k^{re})^2$, deemed as representative of the interval $(\hat{V}_k^{re,r})^2 \leq \mu_k^{re} \leq (\hat{V}_k^{re,r+1})^2$. These parameters need to be defined only for $r \in \{\Gamma^{re} \setminus \{|\Gamma^{re}|\}\}$.

Due to the specific characteristics of the distribution system, the decision variable V_k^{re} may only assume positive values for all buses in the system. Keeping this in mind, it is possible to define the following *tight* values for the disjunctive constants employed above:

$$M_k^{r,re,l} = \tilde{\mu}_k^{re,1} - \tilde{\mu}_k^{re,r} \quad , \forall r \in \{\Gamma^{re} \setminus \{|\Gamma^{re}|\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (368)$$

$$M_k^{r,re,u} = \tilde{\mu}_k^{re,(|\Gamma^{re}|-1)} - \tilde{\mu}_k^{re,r} \quad , \forall r \in \{\Gamma^{re} \setminus \{|\Gamma^{re}|\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (369)$$

Now, the approximation of the function $\mu_k^{im} = (V_k^{im})^2$ is dealt with. A SSPL approximation will also be constructed for this function. Analogously to what has been done above, it is first necessary to discover *which partition of the domain* corresponds to the value of the decision variable V_k^{im} (the argument of the non-linear function). This is done with help of the following constraints:

$$\sum_{s \in \Gamma^{im}} \lambda_k^{im,s} \cdot \hat{V}_k^{im,s} = V_k^{im} \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (370)$$

$$\sum_{s \in \Gamma^{im}} \lambda_k^{im,s} = 1 \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (371)$$

$$\sum_{s \in \Gamma^{im}} y_k^s = 1 \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (372)$$

$$\lambda_k^{im,1} \leq y_k^1 \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (373)$$

$$\lambda_k^{im,s} \leq y_k^{s-1} + y_k^s \quad , \forall s \in \{\Gamma^{im} \setminus \{1\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (374)$$

where:

Γ^{im} Set of indices for *evaluation points* $\hat{V}_k^{im,s}$ and associated variables;

$\hat{V}_k^{im,s}$ Evaluation points of imaginary component of voltage at bus k ;

$\lambda_k^{im,s}$ Weights for expressing the value of the *argument* V_k^{im} as an affine combination of the *evaluation points*;

y_k^s Auxiliary, binary decision variable.

It is possible to use the information of the auxiliary variables y_k^s to check in which partition of the domain the variable V_k^{im} is:

- If $y_k^s = 1$, with $s \in \{\Gamma^{im} \setminus \{|\Gamma^{im}|\}\}$, V_k^{im} is within the partition defined by the interval $\hat{V}_k^{im,s} \leq V_k^{im} \leq \hat{V}_k^{im,s+1}$;
- If $y_k^{|\Gamma^{im}|} = 1$, then $V_k^{im} = \hat{V}_k^{im,|\Gamma^{im}|}$.

With this information at hand, it is possible to use the following disjunctive constraints to construct a SSPL approximation of the function $\mu_k^{im} = (V_k^{im})^2$:

$$M_k^{s,im,l} \cdot (1 - y_k^s) \leq \mu_k^{im} - \tilde{\mu}_k^{im,s} \leq M_k^{s,im,u} \cdot (1 - y_k^s) \\ \forall s \in \{\Gamma^{im} \setminus \{(|\Gamma^{im}| - 1), |\Gamma^{im}|\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (375)$$

$$M_k^{(|\Gamma^{im}|-1),im,l} \cdot (1 - y_k^{(|\Gamma^{im}|-1)} - y_k^{|\Gamma^{im}|}) \leq \mu_k^{im} - \tilde{\mu}_k^{im,(|\Gamma^{im}|-1)} \leq \\ M_k^{(|\Gamma^{im}|-1),im,u} \cdot (1 - y_k^{(|\Gamma^{im}|-1)} - y_k^{|\Gamma^{im}|}) \quad \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (376)$$

where:

$$M_k^{s,im,l}; M_k^{s,im,u}$$

Disjunctive constants. These parameters need to be defined only for $s \in \{\Gamma^{im} \setminus \{|\Gamma^{im}|\}\}$;

$\tilde{\mu}_k^{im,s}$ Representative values of the function $\mu_k^{im} = (V_k^{im})^2$, deemed as representative of the interval $(\hat{V}_k^{im,s})^2 \leq \tilde{\mu}_k^{im,s} \leq (\hat{V}_k^{im,s+1})^2$. These parameters need to be defined only for $s \in \{\Gamma^{im} \setminus \{|\Gamma^{im}|\}\}$.

Keeping in mind that the decision variable V_k^{im} may assume negative and positive values (and also the value zero), but that the function $(V_k^{im})^2$ may only assume non-negative values, it is possible to define the following *tight* values for the disjunctive constants:

$$M_k^{s,im,l} = 0 - \tilde{\mu}_k^{im,s} \quad , \forall s \in \{\Gamma^{im} \setminus \{|\Gamma^{im}|\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (377)$$

$$M_k^{s,im,u} = \max\{\tilde{\mu}_k^{im,1}, \tilde{\mu}_k^{im,(|\Gamma^{im}|-1)}\} - \tilde{\mu}_k^{im,s} \quad , \forall s \in \{\Gamma^{im} \setminus \{|\Gamma^{im}|\}\}, k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (378)$$

Finally, having obtained the values of μ_k^{re} and μ_k^{im} , bounds on bus voltage magnitudes can be enforced with help of the following set of constraints:

$$(\underline{V}_k)^2 \leq \mu_k^{re} + \mu_k^{im} \leq (\bar{V}_k)^2 \quad , \forall k \in \{\Omega_B \setminus \Omega_{REF}\} \quad (379)$$

8.2.1.2.2 Bounds on the magnitude of branch currents

In section 4.2.1.4.2 of chapter 4, bounds on branch current magnitudes have been indirectly enforced by imposing bounds on the square root of the sum of the squared values of the decision variables ι_{km}^{re} and ι_{km}^{im} , which have been defined so as to be at least as high as I_{km}^{re} and I_{km}^{im} , respectively.

For the alternative MILP reformulation of the ACOPF in distribution systems, it is possible to defined an auxiliary decision variable, ϖ_{km} , such that $\varpi_{km} = (\iota_{km})^2$. As ι_{km} may only assume non-negative values, and as the square function is strictly monotonically increasing in the non-negative domain, ensuring that $\varpi_{km} \leq (\bar{I}_{km})^2$ is the same as ensuring that $\iota_{km} \leq \bar{I}_{km}$.

Also, it is possible to define ϖ_{km} as:

$$\varpi_{km} = \varpi_{km}^{re} + \varpi_{km}^{im} \quad , \forall km \in \Psi_C \quad (380)$$

$$\varpi_{km}^{re} = (\iota_{km}^{re})^2 \quad , \forall km \in \Psi_C \quad (381)$$

$$\varpi_{km}^{im} = (\iota_{km}^{im})^2 \quad , \forall km \in \Psi_C \quad (382)$$

It is clear that both ϖ_{km}^{re} and ϖ_{km}^{im} are non-linear functions of a single variable. Thus, for the alternative formulation presented in this chapter, it is possible to construct SSPL approximations of these two functions, by writing equations analogous to those

indicated in section 8.2.1.2.1. For the sake of conciseness, these equations, which are entirely analogous to the ones employed in section 8.2.1.2.1 to approximate $\mu_k^{re} = (V_k^{re})^2$ and $\mu_k^{im} = (V_k^{im})^2$, will not be presented here.

After obtaining the SSPL approximations of ϖ_{km}^{re} and ϖ_{km}^{im} , the branch current magnitude is bounded with help of the following set of constraints:

$$\varpi_{km}^{re} + \varpi_{km}^{im} \leq (\bar{I}_{km})^2, \forall km \in \Psi_C \quad (383)$$

8.2.1.2.3 Bounds on active and reactive power output of generators

The alternative formulation requires no modifications to the constraints presented in subsection 4.2.1.4.3, which may thus be promptly incorporated to the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.1.3 Loads

8.2.1.3.1 Constant-power loads that cannot be shed

Before presenting the linearized equations to be incorporated to the alternative MILP reformulation of the ACOPF for distribution systems, it is worth presenting an alternative formulation of the corresponding non-linear equations, in order to provide the reader with a better comprehension of the reformulation procedure employed here. In the following, equations (9) and (10) of section 2.2.1.3.1 are rewritten, with the substitution of the auxiliary variables ξ_k and ζ_k by the corresponding functions of V_k^{re} and V_k^{im} :

$$I_{d,k}^{re} = (V_k^{re} \cdot d_k^p + V_k^{im} \cdot d_k^q) / (V_k^{re2} + V_k^{im2}), \forall k \in \Omega_{PCTE} \quad (384)$$

$$I_{d,k}^{im} = (V_k^{im} \cdot d_k^p - V_k^{re} \cdot d_k^q) / (V_k^{re2} + V_k^{im2}), \forall k \in \Omega_{PCTE} \quad (385)$$

By substituting $(V_k^{re})^2 = \mu_k^{re}$ and $(V_k^{im})^2 = \mu_k^{im}$ in the above equations and manipulating the expressions algebraically, we obtain the following, still non-linear, equations:

$$I_{d,k}^{re} \cdot \mu_k^{re} + I_{d,k}^{re} \cdot \mu_k^{im} = V_k^{re} \cdot d_k^P + V_k^{im} \cdot d_k^Q \quad , \forall k \in \Omega_{PCTE} \quad (386)$$

$$I_{d,k}^{im} \cdot \mu_k^{re} + I_{d,k}^{im} \cdot \mu_k^{im} = V_k^{im} \cdot d_k^P - V_k^{re} \cdot d_k^Q \quad , \forall k \in \Omega_{PCTE} \quad (387)$$

The products of decision variables $f_{d,k}^{re,re} = I_{d,k}^{re} \cdot \mu_k^{re}$, $f_{d,k}^{re,im} = I_{d,k}^{re} \cdot \mu_k^{im}$, $f_{d,k}^{im,re} = I_{d,k}^{im} \cdot \mu_k^{re}$ and $f_{d,k}^{im,im} = I_{d,k}^{im} \cdot \mu_k^{im}$ need to be reformulated before the equations above can be incorporated into the alternative MILP formulation. For the reformulation of these products, we may take advantage of the fact that the variables μ_k^{re} and μ_k^{im} assume only values in a discretized values, which correspond to the *representative values* of the SSPL approximation described in section 8.2.1.2.1. Thus, the products above may be interpreted as products of a continuous variable (the current component) by a constant (the value assumed by μ_k^{re} or μ_k^{im}). However, it is important to notice that this may assume different *representative values* (the *representative values* of the SSPL approximation for μ_k^{re} and μ_k^{im} , as described in section 8.2.1.2.1), depending on the partition of the domain in which the variables V_k^{re} and V_k^{im} are. Clearly, we have once again disjunctions (partitions) of the decision space.

Thus, the following set of disjunctive constraints may be used to define $f_{d,k}^{re,re}$, $f_{d,k}^{re,im}$, $f_{d,k}^{im,re}$ and $f_{d,k}^{im,im}$:

$$B_k^{r,re,re,l} \cdot (1 - x_k^r) \leq f_{d,k}^{re,re} - I_{d,k}^{re} \cdot \tilde{\mu}_k^{re,r} \leq B_k^{r,re,re,u} \cdot (1 - x_k^r) \\ , \forall r \in \{\Gamma^{re} \setminus \{(|\Gamma^{re}| - 1), |\Gamma^{re}|\}\}, k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (388)$$

$$B_k^{(|\Gamma^{re}|-1),re,re,l} \cdot \left(1 - x_k^{(|\Gamma^{re}|-1)} - x_k^{|\Gamma^{re}|}\right) \leq f_{d,k}^{re,re} - I_{d,k}^{re} \cdot \tilde{\mu}_k^{re,(|\Gamma^{re}|-1)} \leq \\ B_k^{(|\Gamma^{re}|-1),re,re,u} \cdot \left(1 - x_k^{(|\Gamma^{re}|-1)} - x_k^{|\Gamma^{re}|}\right) \quad , \forall k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (389)$$

$$B_k^{s,re,im,l} \cdot (1 - y_k^s) \leq f_{d,k}^{re,im} - I_{d,k}^{re} \cdot \tilde{\mu}_k^{im,s} \leq B_k^{s,re,im,u} \cdot (1 - y_k^s) \\ , \forall s \in \{\Gamma^{im} \setminus \{(|\Gamma^{im}| - 1), |\Gamma^{im}|\}\}, k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (390)$$

$$B_k^{(|\Gamma^{im}|-1),re,im,l} \cdot \left(1 - y_k^{(|\Gamma^{im}|-1)} - y_k^{|\Gamma^{im}|}\right) \leq f_{d,k}^{re,im} - I_{d,k}^{re} \cdot \tilde{\mu}_k^{im,(|\Gamma^{im}|-1)} \leq \\ B_k^{(|\Gamma^{im}|-1),re,im,l} \cdot \left(1 - y_k^{(|\Gamma^{im}|-1)} - y_k^{|\Gamma^{im}|}\right) \quad , \forall k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (391)$$

$$B_k^{r,im,re,l} \cdot (1 - x_k^r) \leq f_{d,k}^{im,re} - I_{d,k}^{im} \cdot \tilde{\mu}_k^{re,r} \leq B_k^{r,im,re,u} \cdot (1 - x_k^r) \\ , \forall r \in \{\Gamma^{re} \setminus \{(|\Gamma^{re}| - 1), |\Gamma^{re}|\}\}, k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (392)$$

$$B_k^{(|\Gamma^{re}|-1),im,re,l} \cdot \left(1 - x_k^{(|\Gamma^{re}|-1)} - x_k^{|\Gamma^{re}|}\right) \leq f_{d,k}^{im,re} - I_{d,k}^{im} \cdot \tilde{\mu}_k^{re, (|\Gamma^{re}|-1)} \leq B_k^{(|\Gamma^{re}|-1),im,re,u} \cdot \left(1 - x_k^{(|\Gamma^{re}|-1)} - x_k^{|\Gamma^{re}|}\right), \forall k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (393)$$

$$B_k^{s,im,im,l} \cdot (1 - y_k^s) \leq f_{d,k}^{im,im} - I_{d,k}^{im} \cdot \tilde{\mu}_k^{im,s} \leq B_k^{s,re,im,u} \cdot (1 - y_k^s), \forall s \in \{\Gamma^{im} \setminus \{(|\Gamma^{im}| - 1), |\Gamma^{im}|\}\}, k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (394)$$

$$B_k^{(|\Gamma^{im}|-1),im,im,l} \cdot \left(1 - y_k^{(|\Gamma^{im}|-1)} - y_k^{|\Gamma^{im}|}\right) \leq f_{d,k}^{im,im} - I_{d,k}^{im} \cdot \tilde{\mu}_k^{im, (|\Gamma^{im}|-1)} \leq B_k^{(|\Gamma^{im}|-1),im,im,l} \cdot \left(1 - y_k^{(|\Gamma^{im}|-1)} - y_k^{|\Gamma^{im}|}\right), \forall k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (395)$$

where:

$$B_k^{r,re,re,l}; B_k^{r,re,re,u}; B_k^{s,re,im,l}; B_k^{s,re,im,u}; B_k^{r,im,re,l}; B_k^{r,im,re,u}; B_k^{s,im,im,l}; B_k^{s,im,im,u}$$

Disjunctive constants.

$$f_{d,k}^{re,re}; f_{d,k}^{re,im}; f_{d,k}^{im,re}; f_{d,k}^{im,im}$$

Auxiliary, continuous variables used for approximating the products

$$f_{d,k}^{re,re} = I_{d,k}^{re} \cdot \mu_k^{re}, \quad f_{d,k}^{re,im} = I_{d,k}^{re} \cdot \mu_k^{im}, \quad f_{d,k}^{im,re} = I_{d,k}^{im} \cdot \mu_k^{re} \quad \text{and} \\ f_{d,k}^{im,im} = I_{d,k}^{im} \cdot \mu_k^{im}.$$

The definition of the disjunctive constants introduced above will not be dealt with here, for the sake of conciseness. Having defined the auxiliary variables $f_{d,k}^{re,re}$, $f_{d,k}^{re,im}$, $f_{d,k}^{im,re}$, and $f_{d,k}^{im,im}$, the constraints through which the current injections from constant-power loads that cannot be shed are related to the associated power injections may be written as:

$$f_{d,k}^{re,re} + f_{d,k}^{re,im} = V_k^{re} \cdot d_k^P + V_k^{im} \cdot d_k^Q, \forall k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (396)$$

$$f_{d,k}^{im,re} + f_{d,k}^{im,im} = V_k^{im} \cdot d_k^P - V_k^{re} \cdot d_k^Q, \forall k \in \{\Omega_{PCTE} \setminus \Omega_{SHED}\} \quad (397)$$

8.2.1.3.2 Constant-power loads that can be shed

As indicated in chapter 4, load shedding is considered to be a discrete decision: the load at bus k will be considered to be either energized ($\rho_k = 0$) or de-energized

($\rho_k = 1$). When the load at bus k is shed, it is obviously necessary to ensure that the current components $I_{d,k}^{re}$ and $I_{d,k}^{im}$ will be forcefully set to zero.

In order to do that and still be able to use constraints similar to these employed in section 8.2.1.3.1 in the alternative formulation, the auxiliary variables $I_{l,k}^{re}$ and $I_{l,k}^{im}$, which correspond to the values of the load currents “before load shedding is taken into account”, are defined. Also, disjunctive constraints that ensure that $I_{d,k}^{re} = I_{l,k}^{re}$ and $I_{d,k}^{im} = I_{l,k}^{im}$ when $\rho_k = 0$, but that $I_{d,k}^{re} = 0$ and $I_{d,k}^{im} = 0$ when $\rho_k = 1$, will be introduced to the alternative MILP formulation. Before doing that it is necessary to deal with the definition of the auxiliary variables $I_{l,k}^{re}$ and $I_{l,k}^{im}$.

In order to define the auxiliary variables $I_{l,k}^{re}$ and $I_{l,k}^{im}$, a procedure similar to that employed in section 8.2.1.3.1 for the definition of $I_{d,k}^{re}$ and $I_{d,k}^{im}$ will be used. That is to say, for each $k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\}$, the auxiliary variables $f_{l,k}^{re,re}$, $f_{l,k}^{re,im}$, $f_{l,k}^{im,re}$ and $f_{l,k}^{im,im}$ will be defined. These auxiliary decision variables will be used to approximate the products $f_{l,k}^{re,re} = I_{l,k}^{re} \cdot \mu_k^{re}$, $f_{l,k}^{re,im} = I_{l,k}^{re} \cdot \mu_k^{im}$, $f_{l,k}^{im,re} = I_{l,k}^{im} \cdot \mu_k^{re}$ and $f_{l,k}^{im,im} = I_{l,k}^{im} \cdot \mu_k^{im}$ – analogously to what has been done for the products $f_{d,k}^{re,re} = I_{d,k}^{re} \cdot \mu_k^{re}$, $f_{d,k}^{re,im} = I_{d,k}^{re} \cdot \mu_k^{im}$, $f_{d,k}^{im,re} = I_{d,k}^{im} \cdot \mu_k^{re}$ and $f_{d,k}^{im,im} = I_{d,k}^{im} \cdot \mu_k^{im}$ in equations (388) to (395). The constraints used for the definition of the auxiliary variables $f_{l,k}^{re,re}$, $f_{l,k}^{re,im}$, $f_{l,k}^{im,re}$ and $f_{l,k}^{im,im}$ will not be written here, as they are absolutely analogous to equations (388) to (395).

Then, the following constraints will implicitly relate the values of $I_{l,k}^{re}$ and $I_{l,k}^{im}$ to the power injections of constant-power loads that can be shed:

$$f_{l,k}^{re,re} + f_{l,k}^{re,im} = V_k^{re} \cdot d_k^P + V_k^{im} \cdot d_k^Q, \quad \forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (398)$$

$$f_{l,k}^{im,re} + f_{l,k}^{im,im} = V_k^{im} \cdot d_k^P - V_k^{re} \cdot d_k^Q, \quad \forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (399)$$

where:

$$f_{l,k}^{re,re} ; f_{l,k}^{re,im} ; f_{l,k}^{im,re} ; f_{l,k}^{im,im}$$

Auxiliary, continuous decision variables used for approximating the products

$$f_{l,k}^{re,re} = I_{l,k}^{re} \cdot \mu_k^{re}, \quad f_{l,k}^{re,im} = I_{l,k}^{re} \cdot \mu_k^{im}, \quad f_{l,k}^{im,re} = I_{l,k}^{im} \cdot \mu_k^{re} \quad \text{and} \quad f_{l,k}^{im,im} = I_{l,k}^{im} \cdot \mu_k^{im}.$$

Finally, the following disjunctive constraints ensure that $I_{d,k}^{re} = I_{l,k}^{re}$ and $I_{d,k}^{im} = I_{l,k}^{im}$ when $\rho_k = 0$, but that $I_{d,k}^{re} = 0$ and $I_{d,k}^{im} = 0$ when $\rho_k = 1$:

$$M_k^{D,re,PCTE,1} \cdot \rho_k \leq I_{d,k}^{re} - I_{l,k}^{re} \leq M_k^{D,re,PCTE,2} \cdot \rho_k, \forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (400)$$

$$M_k^{D,re,PCTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{re} \leq M_k^{D,re,PCTE,4} \cdot (1 - \rho_k), \forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (401)$$

$$M_k^{D,im,PCTE,1} \cdot \rho_k \leq I_{d,k}^{im} - I_{l,k}^{im} \leq M_k^{D,im,PCTE,2} \cdot \rho_k, \forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (402)$$

$$M_k^{D,im,PCTE,3} \cdot (1 - \rho_k) \leq I_{d,k}^{im} \leq M_k^{D,im,PCTE,4} \cdot (1 - \rho_k), \forall k \in \{\Omega_{PCTE} \cap \Omega_{SHED}\} \quad (403)$$

where:

$$M_k^{D,re,PCTE,1} ; M_k^{D,re,PCTE,2} ; M_k^{D,re,PCTE,3} ; M_k^{D,re,PCTE,4}$$

$$M_k^{D,im,PCTE,1} ; M_k^{D,im,PCTE,2} ; M_k^{D,im,PCTE,3} ; M_k^{D,im,PCTE,4}$$

Disjunctive constants, whose definition will not be dealt with here, for the sake of conciseness.

8.2.1.3.3 Constant-current loads that cannot be shed

The investigation of the alternative MILP reformulation presented in this chapter has been interrupted before the treatment of loads of the constant-current type, and therefore no definition of constraints for obtaining the current injections corresponding to these types of loads is currently available.

8.2.1.3.4 Constant-current loads that can be shed

The investigation of the alternative MILP reformulation presented in this chapter has been interrupted before the treatment of loads of the constant-current type, and therefore no definition of constraints for obtaining the current injections corresponding to these types of loads is currently available.

8.2.1.3.5 Constant-impedance loads that cannot be shed

The alternative formulation requires no modifications to the constraints presented in subsection 4.2.1.3.5, which may thus be promptly incorporated to the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.1.3.6 Constant-impedance loads that can be shed

The alternative formulation requires no modifications to the constraints presented in subsection 4.2.1.3.6, which may thus be promptly incorporated to the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.1.4 Generation

8.2.1.4.1 Non-curtailable generators with no control over the active power output

Similarly to what has been done for loads, the first step for obtaining the constraints that will be used in the alternative MILP reformulation of the ACOPF is to rewrite the original, non-linear equations that relate the current injections with the active and reactive power output of generators, substituting the auxiliary variables ξ_k and ζ_k by the corresponding functions of V_k^{re} and V_k^{im} :

$$I_{g,k}^{re} = (V_k^{re} \cdot g_k^P + V_k^{im} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad , \forall k \in \Omega_{GEN} \quad (404)$$

$$I_{g,k}^{im} = (V_k^{im} \cdot g_k^P - V_k^{re} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad , \forall k \in \Omega_{GEN} \quad (405)$$

By substituting $(V_k^{re})^2 = \mu_k^{re}$ and $(V_k^{im})^2 = \mu_k^{im}$ in the above equations and manipulating the expressions algebraically, we obtain the following, still non-linear equations:

$$I_{g,k}^{re} \cdot \mu_k^{re} + I_{g,k}^{re} \cdot \mu_k^{im} = V_k^{re} \cdot g_k^P + V_k^{im} \cdot g_k^Q \quad , \forall k \in \Omega_{GEN} \quad (406)$$

$$I_{g,k}^{im} \cdot \mu_k^{re} + I_{g,k}^{im} \cdot \mu_k^{im} = V_k^{im} \cdot g_k^P - V_k^{re} \cdot g_k^Q \quad , \forall k \in \Omega_{GEN} \quad (407)$$

The products of decision variables $f_{g,k}^{re,re} = I_{g,k}^{re} \cdot \mu_k^{re}$, $f_{g,k}^{re,im} = I_{g,k}^{re} \cdot \mu_k^{im}$, $f_{g,k}^{im,re} = I_{g,k}^{im} \cdot \mu_k^{re}$ and $f_{g,k}^{im,im} = I_{g,k}^{im} \cdot \mu_k^{im}$ need to be reformulated before the previous equations can be incorporated into the alternative MILP formulation. For the reformulation of these products, it suffices to define constraints analogous to those represented by equations (388) to (395). Then, the following constraints will implicitly relate the values of $I_{g,k}^{re}$ and $I_{g,k}^{im}$ to the power injections of constant-power loads that can be shed:

$$f_{g,k}^{re,re} + f_{g,k}^{re,im} = V_k^{re} \cdot g_k^P + V_k^{im} \cdot g_k^Q \quad , \forall k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\} \quad (408)$$

$$f_{g,k}^{im,re} + f_{g,k}^{im,im} = V_k^{im} \cdot g_k^P - V_k^{re} \cdot g_k^Q \quad , \forall k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\} \quad (409)$$

The reader will notice that equations (408) and (409) are still nonlinear, due to the products of decision variables¹⁴ $V_k^{im} \cdot g_k^Q$ and $V_k^{re} \cdot g_k^Q$. In order to reformulate these equations and allow their incorporation to the alternative MILP formulation of the ACOFP for distribution systems, these products are substituted respectively by the auxiliary decision variables $m_k^{Q,re}$ and $m_k^{Q,im}$, and the equations (408) and (409) are reformulated as:

$$f_{g,k}^{re,re} + f_{g,k}^{re,im} = V_k^{re} \cdot g_k^P + m_k^{Q,re} \quad , \forall k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\} \quad (410)$$

$$f_{g,k}^{im,re} + f_{g,k}^{im,im} = V_k^{im} \cdot g_k^P - m_k^{Q,im} \quad , \forall k \in \{\Omega_{CTRQ} \setminus \Omega_{CURT}\} \quad (411)$$

where:

$$f_{g,k}^{re,re} ; f_{g,k}^{re,im} ; f_{g,k}^{im,re} ; f_{g,k}^{im,im}$$

Auxiliary, continuous variables used for approximating the products

$$f_{g,k}^{re,re} = I_{g,k}^{re} \cdot \mu_k^{re}, \quad f_{g,k}^{re,im} = I_{g,k}^{re} \cdot \mu_k^{im}, \quad f_{g,k}^{im,re} = I_{g,k}^{im} \cdot \mu_k^{re} \quad \text{and}$$

$$f_{g,k}^{im,im} = I_{g,k}^{im} \cdot \mu_k^{im}.$$

$m_k^{Q,re}$ Auxiliary decision variable for modeling the product $V_k^{im} \cdot g_k^Q$;

$m_k^{Q,im}$ Auxiliary decision variable for modeling the product $V_k^{re} \cdot g_k^Q$.

¹⁴ It is important to keep in mind that, for generators in $\{\Omega_{CTRQ} \setminus \Omega_{CURT}\}$, the active power output is a parameter of the optimization problem.

Finally, it is necessary to define the linear constraints through which the auxiliary variables $m_k^{Q,re}$ and $m_k^{Q,im}$ are bounded within the convex envelopes (more precisely, McCormick's envelopes) of the original products:

$$m_k^{Q,re} \geq V_k^{im,min} \cdot g_k^Q + V_k^{im} \cdot \underline{g}_k^Q - V_k^{im,min} \cdot \underline{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (412)$$

$$m_k^{Q,re} \geq V_k^{im,max} \cdot g_k^Q + V_k^{im} \cdot \bar{g}_k^Q - V_k^{im,max} \cdot \bar{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (413)$$

$$m_k^{Q,re} \leq V_k^{im,min} \cdot g_k^Q + V_k^{im} \cdot \bar{g}_k^Q - V_k^{im,min} \cdot \bar{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (414)$$

$$m_k^{Q,re} \leq V_k^{im,max} \cdot g_k^Q + V_k^{im} \cdot \underline{g}_k^Q - V_k^{im,max} \cdot \underline{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (415)$$

$$m_k^{Q,im} \geq V_k^{re,min} \cdot g_k^Q + V_k^{re} \cdot \underline{g}_k^Q - V_k^{re,min} \cdot \underline{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (416)$$

$$m_k^{Q,im} \geq V_k^{re,max} \cdot g_k^Q + V_k^{re} \cdot \bar{g}_k^Q - V_k^{re,max} \cdot \bar{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (417)$$

$$m_k^{Q,im} \leq V_k^{re,min} \cdot g_k^Q + V_k^{re} \cdot \bar{g}_k^Q - V_k^{re,min} \cdot \bar{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (418)$$

$$m_k^{Q,im} \leq V_k^{re,max} \cdot g_k^Q + V_k^{re} \cdot \underline{g}_k^Q - V_k^{re,max} \cdot \underline{g}_k^Q \quad , \forall k \in \Omega_{GEN} \quad (419)$$

8.2.1.4.2 Curtailable generators with no control over the active power output

As indicated in chapter 4, generation curtailment is considered to be a discrete decision in the proposed formulation: the generator at bus k will be considered to be either energized ($\tau_k = 0$) or de-energized ($\tau_k = 1$). Therefore, it is necessary to ensure that, if the generator connected to bus k is curtailed, $I_{g,k}^{re}$ and $I_{g,k}^{im}$ will be forcefully set to zero.

In order to do that, the auxiliary variables $I_{p,k}^{re}$ and $I_{p,k}^{im}$, which correspond to the values of the generator currents “before generation curtailment is taken into account”, are defined. Also, disjunctive constraints that ensure that $I_{g,k}^{re} = I_{p,k}^{re}$ and $I_{g,k}^{im} = I_{p,k}^{im}$ when $\tau_k = 0$, but that $I_{g,k}^{re} = 0$ and $I_{g,k}^{im} = 0$ when $\tau_k = 1$, will be incorporated to the alternative MILP formulation. Before doing that, it is necessary to deal with the definition of the auxiliary variables $I_{p,k}^{re}$ and $I_{p,k}^{im}$.

This is done by defining the auxiliary variables $f_{p,k}^{re,re}$, $f_{p,k}^{re,im}$, $f_{p,k}^{im,re}$ and $f_{p,k}^{im,im}$, and utilizing equations analogous to (388) to (395) to ensure that these auxiliary

variables correspond, respectively, to approximations of the products $I_{p,k}^{re} \cdot \mu_k^{re}$, $I_{p,k}^{re} \cdot \mu_k^{im}$, $I_{p,k}^{im} \cdot \mu_k^{re}$ and $I_{p,k}^{im} \cdot \mu_k^{im}$. After that, the following equations are defined:

$$f_{p,k}^{re,re} + f_{p,k}^{re,im} = V_k^{re} \cdot g_k^p + m_k^{Q,re}, \quad \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (420)$$

$$f_{p,k}^{im,re} + f_{p,k}^{im,im} = V_k^{im} \cdot g_k^p - m_k^{Q,im}, \quad \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (421)$$

where the auxiliary decision variables $m_k^{Q,re}$ and $m_k^{Q,im}$ have already been defined – see equations (412) to (419).

Finally, the following disjunctive constraints ensure that $I_{g,k}^{re} = I_{p,k}^{re}$ and $I_{g,k}^{im} = I_{p,k}^{im}$ when $\tau_k = 0$, but that $I_{g,k}^{re} = 0$ and $I_{g,k}^{im} = 0$ when $\tau_k = 1$:

$$M_k^{G,re,1} \cdot \tau_k \leq I_{g,k}^{re} - I_{p,k}^{re} \leq M_k^{G,re,2} \cdot \tau_k, \quad \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (422)$$

$$M_k^{G,re,3} \cdot (1 - \tau_k) \leq I_{g,k}^{re} \leq M_k^{G,re,4} \cdot (1 - \tau_k), \quad \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (423)$$

$$M_k^{G,im,1} \cdot \tau_k \leq I_{g,k}^{im} - I_{p,k}^{im} \leq M_k^{G,im,2} \cdot \tau_k, \quad \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (424)$$

$$M_k^{G,im,3} \cdot (1 - \tau_k) \leq I_{g,k}^{im} \leq M_k^{G,im,4} \cdot (1 - \tau_k), \quad \forall k \in \{\Omega_{CTRQ} \cap \Omega_{CURT}\} \quad (425)$$

where:

$$M_k^{G,re,1}; M_k^{G,re,2}; M_k^{G,re,3}; M_k^{G,re,4}; M_k^{G,im,1}; M_k^{G,im,2}; M_k^{G,im,3}; M_k^{G,im,4}$$

Disjunctive constants, whose definition will not be dealt with here, for the sake of conciseness.

8.2.1.4.3 Generators with control over the active power output

In order to model these generators, it suffices to define the auxiliary variables $f_{g,k}^{re,re}$, $f_{g,k}^{re,im}$, $f_{g,k}^{im,re}$ and $f_{g,k}^{im,im}$ and use equations analogous to (388) to (395) to ensure that these auxiliary variables correspond, respectively, to approximations of the

products $I_{g,k}^{re} \cdot \mu_k^{re}$, $I_{g,k}^{re} \cdot \mu_k^{im}$, $I_{g,k}^{im} \cdot \mu_k^{re}$ and $I_{g,k}^{im} \cdot \mu_k^{im}$. After that, the following equations are defined:

$$f_{g,k}^{re,re} + f_{g,k}^{re,im} = m_k^{P,re} + m_k^{Q,re} \quad , \forall k \in \Omega_{CTRPQ} \quad (426)$$

$$f_{g,k}^{im,re} + f_{g,k}^{im,im} = m_k^{P,im} - m_k^{Q,im} \quad , \forall k \in \Omega_{CTRPQ} \quad (427)$$

where the auxiliary variables $m_k^{Q,re}$ and $m_k^{Q,im}$ have already been defined, and:

$m_k^{P,re}$ Auxiliary decision variable for modeling the product $V_k^{re} \cdot g_k^P$;

$m_k^{P,im}$ Auxiliary decision variable for modeling the product $V_k^{im} \cdot g_k^P$.

The following constraints are then employed to bound the auxiliary decision variables $m_k^{P,re}$ and $m_k^{P,im}$ within the convex envelope of the original products:

$$m_k^{P,re} \geq V_k^{re,min} \cdot g_k^P + V_k^{re} \cdot \underline{g}_k^P - V_k^{re,min} \cdot \underline{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (428)$$

$$m_k^{P,re} \geq V_k^{re,max} \cdot g_k^P + V_k^{re} \cdot \bar{g}_k^P - V_k^{re,max} \cdot \bar{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (429)$$

$$m_k^{P,re} \leq V_k^{re,min} \cdot g_k^P + V_k^{re} \cdot \bar{g}_k^P - V_k^{re,min} \cdot \bar{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (430)$$

$$m_k^{P,re} \leq V_k^{re,max} \cdot g_k^P + V_k^{re} \cdot \underline{g}_k^P - V_k^{re,max} \cdot \underline{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (431)$$

$$m_k^{P,im} \geq V_k^{im,min} \cdot g_k^P + V_k^{im} \cdot \underline{g}_k^P - V_k^{im,min} \cdot \underline{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (432)$$

$$m_k^{P,im} \geq V_k^{im,max} \cdot g_k^P + V_k^{im} \cdot \bar{g}_k^P - V_k^{im,max} \cdot \bar{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (433)$$

$$m_k^{P,im} \leq V_k^{im,min} \cdot g_k^P + V_k^{im} \cdot \bar{g}_k^P - V_k^{im,min} \cdot \bar{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (434)$$

$$m_k^{P,im} \leq V_k^{im,max} \cdot g_k^P + V_k^{im} \cdot \underline{g}_k^P - V_k^{im,max} \cdot \underline{g}_k^P \quad , \forall k \in \Omega_{CTRPQ} \quad (435)$$

8.2.1.5 Voltage reference buses

The alternative formulation requires no modifications to the constraints (145)-(148) presented in subsection 4.2.1.5, which may thus be promptly incorporated to the alternative MILP reformulation of the ACOPT in distribution systems.

8.2.1.6 *Slack buses and buses without generators and/or loads*

The alternative formulation requires no modifications to the constraints presented in subsection 4.2.1.6, which may thus be promptly incorporated to the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.1.7 *Radiality constraints*

The alternative formulation requires no modifications to the constraints presented in subsection 4.2.1.7, which may thus be promptly incorporated to the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.2 Objective functions for selected distribution system operations and expansion planning applications

8.2.2.1 *Minimization of costs of load shedding*

The alternative formulation requires no modifications to the objective function presented in subsection 4.2.2.1, which can therefore be promptly employed with the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.2.2 *Minimization of curtailment of non-controllable generation*

The alternative formulation requires no modifications to the objective function presented in subsection 4.2.2.2, which can therefore be promptly employed with the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.2.3 *Minimization of generation costs*

The alternative formulation requires no modifications to the objective function presented in subsection 4.2.2.3, which can therefore be promptly employed with the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.2.4 *Minimization of costs of power imports*

If the magnitudes of the bus voltages at the buses at the interface with the external system, $V_k^{ref} \forall k \in \Omega_{ITFC}$, are parameters of the optimization problem, equation (199) of section 4.2.2.4 may be promptly used in the alternative MILP reformulation of the ACOPF for distribution systems.

However, if this voltage magnitude is to be considered as a decision variable, it would be required to obtain approximations of the decision variables $V_k, \forall k \in \Omega_{ITFC}$, before modeling the objective function related to the minimization of the costs of power imports. The reader will notice that, in the alternative MILP formulation presented in this chapter, the bounds on bus voltage magnitudes were enforced via the constraint $(\underline{V}_k)^2 \leq \mu_k^{re} + \mu_k^{im} \leq (\overline{V}_k)^2$ – see equation (358) of section 8.2.1.2.1. Thus, no approximation of the bus voltage magnitudes, V_k , has yet been defined for the alternative MILP reformulation. The investigation of the alternative MILP reformulation presented in this chapter has been interrupted before any approximations for V_k were defined, and therefore no formulation of the objective function of minimization of the costs of power imports has been defined for the case in which the voltage magnitude at the interfaces with the external system are considered as decision variables.

8.2.2.5 *Minimization of costs of ohmic losses*

In section 4.2.2.5, two alternative formulations of the objective function for the problem of minimization of ohmic losses have been defined.

The formulation of the objective function corresponding to equation (215) may be modified for its use with the alternative MILP reformulation of the ACOPF, as follows:

- It is necessary to recall that the investigation of the alternative MILP reformulation presented in this chapter has been interrupted before constant-current loads had been treated. Therefore, it is necessary to remove the terms that relate to loads of the constant-current type from equation (215).

Also, the formulation of the objective function corresponding to equation (225) may be modified for its use within the alternative MILP reformulation of the ACOPF, as follows:

- First, it is necessary to remove the terms that relate to loads of the constant-current type from equation (225), as the investigation of the alternative MILP reformulation presented in this chapter has been interrupted before these loads had been treated.
- Then, it is necessary to recall the approximation $\mu_k = \mu_k^{re} + \mu_k^{im} \approx V_k^2$ has already been defined for the alternative MILP reformulation presented in this chapter. This term, which is employed in several equations of section 4.2.2.5, will be readily available when the alternative MILP reformulation is used.

8.2.2.6 *Minimization of costs of reinforcements to the distribution system*

The alternative formulation requires no modifications to the objective function presented in subsection 4.2.2.6, which can therefore be promptly employed with the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.2.7 *Minimization of costs of capacitor placement*

The alternative formulation requires no modifications to the objective function presented in subsection 4.2.2.7, which can therefore be promptly employed with the alternative MILP reformulation of the ACOPF in distribution systems.

8.2.2.8 *Minimization of circuit switching costs*

The alternative formulation requires no modifications to the objective function presented in subsection 4.2.2.8, which can therefore be promptly employed with the alternative MILP reformulation of the ACOPF in distribution systems.

9 APPENDIX C: PIECEWISE-LINEAR APPROXIMATIONS OF GENERATOR CURRENTS

Constraints used for obtaining the currents injected into the network by generators have been presented in section 4.2.1.2 of this dissertation. The constraints presented in section 4.2.1.2 employ McCormick's envelopes to reformulate (and approximate) products of two decision variables. As discussed in previous sections of this dissertation, it is not possible to achieve an arbitrarily accurate approximation of bilinear products when McCormick's envelopes are used – the approximation accuracy is implicitly dictated by the bounds on the continuous variables that form the products.

However, it is possible to employ alternative formulations of the constraints used for obtaining the generator currents, completely eliminating the need to employ McCormick's envelopes. This alternative formulation is based in constructing piecewise-linear approximations of the generator currents with help of SOS2. This allows the user to arbitrate the accuracy of the approximation of the generation currents while determining the number and location of the *evaluation points*. However, it should be kept in mind that enhancing the accuracy of the piecewise-linear approximation by augmenting the number of evaluation points may result in additional computational requirements.

The alternative formulation for the constraints used for obtaining the generator currents are presented in the following sections. Section 9.1 deals with the generators with no control over their active power output, whereas generators that do control their active power output are treated in section 9.2.

9.1 Generators with no control over the active power output

The formulation presented below is based on treating the generator currents $I_{g,k}^{re}$ and $I_{g,k}^{im}$ as functions of three decision variables – i.e., $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q)$ and $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q)$ – and then constructing piecewise-linear approximations of these

functions with help of SOS2. In order to better understand why the currents of generators with no control over the active power output may be treated as functions of three (continuous) decision variables, the reader may refer to the following equations, which correspond to equations (75) and (76) of section 4.2.1.2.1:

$$I_{g,k}^{re} = (V_k^{re} \cdot g_k^P + V_k^{im} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (436)$$

$$I_{g,k}^{im} = (V_k^{im} \cdot g_k^P - V_k^{re} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (437)$$

It is clear that, as g_k^P is a fixed value (a parameter) for generators that do not control their active power output, $I_{g,k}^{re}$ and $I_{g,k}^{im}$ are functions of three decision variables. In the following, the constraints used for constructing piecewise-linear approximations of these functions are presented:

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \sum_{i \in T^Q} \varrho_k^{r,s,i} \cdot \begin{bmatrix} \hat{I}_{g,k}^{re,r,s,i} \\ \hat{I}_{g,k}^{im,r,s,i} \end{bmatrix} = \begin{bmatrix} I_{g,k}^{re} \\ I_{g,k}^{im} \end{bmatrix} \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (438)$$

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \sum_{i \in T^Q} \varrho_k^{r,s,i} \cdot \begin{bmatrix} \hat{V}_k^{re,r} \\ \hat{V}_k^{im,s} \\ \hat{g}_k^{Q,i} \end{bmatrix} = \begin{bmatrix} V_k^{re} \\ V_k^{im} \\ g_k^Q \end{bmatrix} \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (439)$$

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \sum_{i \in T^Q} \varrho_k^{r,s,i} = 1 \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (440)$$

where:

- T^Q Set of indices for *evaluation points* $\hat{g}_k^{Q,i}$ and associated variables;
- $\hat{g}_k^{Q,i}$ Evaluation points of reactive power output of generator at bus k ;
- $\hat{I}_{g,k}^{re,r,s,i}$ Evaluated values of function $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q)$, for bus k ;
- $\hat{I}_{g,k}^{im,r,s,i}$ Evaluated values of function $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q)$, for bus k ;
- $\varrho_k^{r,s,i}$ Weights for constructing piecewise-linear approximation of non-convex, non-linear functions.

$$\sum_{r \in \Gamma^{re}} d1_k^r = 1 \quad , \forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (441)$$

$$\varrho_k^{1,s,i} \leq d1_k^1 \quad , \forall s \in \Gamma^{im}, i \in T^Q, k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (442)$$

$$\begin{aligned} \varrho_k^{r,s,i} &\leq d1_k^{r-1} + d1_k^r \\ &,\forall r \in \{\Gamma^{re} \setminus \{1\}\}, s \in \Gamma^{im}, i \in T^Q, k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \end{aligned} \quad (443)$$

$$\sum_{s \in \Gamma^{im}} d2_k^s = 1 \quad ,\forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (444)$$

$$\varrho_k^{r,1,i} \leq d2_k^1 \quad ,\forall r \in \Gamma^{re}, i \in T^Q, k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (445)$$

$$\begin{aligned} \varrho_k^{r,s,i} &\leq d2_k^{s-1} + d2_k^s \\ &,\forall r \in \Gamma^{re}, s \in \{\Gamma^{im} \setminus \{1\}\}, i \in T^Q, k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \end{aligned} \quad (446)$$

$$\sum_{i \in T^Q} d3_k^i = 1 \quad ,\forall k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (447)$$

$$\varrho_k^{r,s,1} \leq d3_k^1 \quad ,\forall r \in \Gamma^{re}, s \in \Gamma^{im}, k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \quad (448)$$

$$\begin{aligned} \varrho_k^{r,s,i} &\leq d3_k^{i-1} + d3_k^i \\ &,\forall r \in \Gamma^{re}, s \in \Gamma^{im}, i \in \{T^Q \setminus \{1\}\}, k \in \{\Omega_{CTRQ} \cap \Omega_{NCRT}\} \end{aligned} \quad (449)$$

where $d1_k^r$, $d2_k^s$ and $d3_k^i$ are auxiliary binary decision variables.

9.2 Generators with control over the active power output

The formulation presented below is based on treating the generator currents $I_{g,k}^{re}$ and $I_{g,k}^{im}$ as functions of four decision variables – i.e., $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$ and $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$ – and then constructing piecewise-linear approximations of these functions with help of SOS2. In order to better understand why the currents of generators with control over the active power output are treated as functions of four decision variables, the reader may refer to the following equations, which correspond to equations (98) and (99) of section 4.2.1.2.3:

$$I_{g,k}^{re} = (V_k^{re} \cdot g_k^P + V_k^{im} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad ,\forall k \in \Omega_{CTRQ} \quad (450)$$

$$I_{g,k}^{im} = (V_k^{im} \cdot g_k^P - V_k^{re} \cdot g_k^Q) / (V_k^{re2} + V_k^{im2}) \quad ,\forall k \in \Omega_{CTRQ} \quad (451)$$

In the following, the constraints used for constructing piecewise-linear approximations of $I_{g,k}^{re}$ and $I_{g,k}^{im}$ are presented:

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \sum_{i \in T^Q} \sum_{j \in T^P} \varrho_k^{r,s,i,j} \cdot \begin{bmatrix} \hat{f}_{g,k}^{re,r,s,i,j} \\ \hat{f}_{g,k}^{im,r,s,i,j} \end{bmatrix} = \begin{bmatrix} I_{g,k}^{re} \\ I_{g,k}^{im} \end{bmatrix}, \forall k \in \Omega_{CTRPQ} \quad (452)$$

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \sum_{i \in T^Q} \sum_{j \in T^P} \varrho_k^{r,s,i,j} \cdot \begin{bmatrix} \hat{V}_k^{re,r} \\ \hat{V}_k^{im,s} \\ \hat{g}_k^{Q,i} \\ \hat{g}_k^{P,j} \end{bmatrix} = \begin{bmatrix} V_k^{re} \\ V_k^{im} \\ g_k^Q \\ g_k^P \end{bmatrix}, \forall k \in \Omega_{CTRPQ} \quad (453)$$

$$\sum_{r \in \Gamma^{re}} \sum_{s \in \Gamma^{im}} \sum_{i \in T^Q} \sum_{j \in T^P} \varrho_k^{r,s,i,j} = 1, \forall k \in \Omega_{CTRPQ} \quad (454)$$

where:

- T^P Set of indices for *evaluation points* $\hat{g}_k^{P,j}$ and associated variables;
- $\hat{g}_k^{P,j}$ Evaluation points of active power output of generator at bus k ;
- $\hat{f}_{g,k}^{re,r,s,i,j}$ Evaluated values of function $I_{g,k}^{re}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$, for bus k ;
- $\hat{f}_{g,k}^{im,r,s,i,j}$ Evaluated values of function $I_{g,k}^{im}(V_k^{re}, V_k^{im}, g_k^Q, g_k^P)$, for bus k ;
- $\varrho_k^{r,s,i,j}$ Weights for constructing piecewise-linear approximation of non-convex, non-linear functions.

$$\sum_{r \in \Gamma^{re}} d1_k^r = 1, \forall k \in \Omega_{CTRPQ} \quad (455)$$

$$\varrho_k^{1,s,i,j} \leq d1_k^1, \forall s \in \Gamma^{im}, i \in T^Q, j \in T^P, k \in \Omega_{CTRPQ} \quad (456)$$

$$\varrho_k^{r,s,i,j} \leq d1_k^{r-1} + d1_k^r, \forall r \in \{\Gamma^{re} \setminus \{1\}\}, s \in \Gamma^{im}, i \in T^Q, j \in T^P, k \in \Omega_{CTRPQ} \quad (457)$$

$$\sum_{s \in \Gamma^{im}} d2_k^s = 1, \forall k \in \Omega_{CTRPQ} \quad (458)$$

$$\varrho_k^{r,1,i,j} \leq d2_k^1, \forall r \in \Gamma^{re}, i \in T^Q, j \in T^P, k \in \Omega_{CTRPQ} \quad (459)$$

$$\varrho_k^{r,s,i,j} \leq d2_k^{s-1} + d2_k^s, \forall r \in \Gamma^{re}, s \in \{\Gamma^{im} \setminus \{1\}\}, i \in T^Q, j \in T^P, k \in \Omega_{CTRPQ} \quad (460)$$

$$\sum_{i \in T^Q} d3_k^i = 1, \forall k \in \Omega_{CTRPQ} \quad (461)$$

$$\varrho_k^{r,s,1,j} \leq d3_k^1, \forall r \in \Gamma^{re}, s \in \Gamma^{im}, j \in T^P, k \in \Omega_{CTRPQ} \quad (462)$$

$$\varrho_k^{r,s,i,j} \leq d3_k^{i-1} + d3_k^i, \forall r \in \Gamma^{re}, s \in \Gamma^{im}, i \in \{T^Q \setminus \{1\}\}, j \in T^P, k \in \Omega_{CTRPQ} \quad (463)$$

$$\sum_{j \in T^P} d4_k^j = 1, \forall k \in \Omega_{CTRPQ} \quad (464)$$

$$q_k^{r,s,i,1} \leq d4_k^1, \forall r \in \Gamma^{re}, s \in \Gamma^{im}, j \in T^P, k \in \Omega_{CTRPQ} \quad (465)$$

$$q_k^{r,s,i,j} \leq d4_k^{j-1} + d4_k^j, \forall r \in \Gamma^{re}, s \in \Gamma^{im}, i \in T^Q, j \in \{T^P \setminus \{1\}\}, k \in \Omega_{CTRPQ} \quad (466)$$

where $d1_k^r$, $d2_k^s$, $d3_k^i$ and $d4_k^j$ are auxiliary binary decision variables.