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CONTROL LYAPUNOV FUNCTION DESIGN OF ADVERTISING IN A DUOPOLY

Bernardo Cardoso de Aquino Cruz

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Examinada por:

Prof. Amit Bhaya, Ph.D.

Prof. Tiago Roux de Oliveira, D. Sc.

Prof. Eugenius Kaszkurewicz, D. Sc.

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Orientador: Amit Bhaya

Programa: Engenharia Elétrica

Neste trabalho estudamos a viabilidade e a adequação ao mundo real de um modelo de mercado de duopólio com churn. Para isso fazemos uso de uma estratégia baseada em CLF para propaganda predatória para estressar o modelo em alguns cenários pré determinados. Inicialmente, o CLF é comparado com um esforço predatório constante, bem como com outra estratégia CLF idêntica, e resultados teóricos e práticos são apresentados. Em seguida, um estimador baseado em um diferenciador robusto é introduzido juntamente com o CLF, e sua eficiência é comparada com a estratégia CLF comum. Subsequentemente, ela é confrontada com um controle idêntico, e novamente, os resultados são observados e estudados. Por último, as limitações encontradas são estudadas e conclusões são obtidas.

Abstract of Dissertation presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Master of Science (M.Sc.)

CONTROL LYAPUNOV FUNCTION DESIGN OF ADVERTISING IN A DUOPOLY

Bernardo Cardoso de Aquino Cruz

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Advisor: Amit Bhaya

Department: Electrical Engineering

In this work we study the feasibility and adequacy to the real world of a duopoly market model with churn. For that we use of a CLF based strategy of predatory advertising to stress the model and observe how it behaves in some pre determined scenarios. Initially, the CLF is compared with constant predation effort, as well as with another identical CLF strategy, and theoretical and practical results are obtained. A robust differentiator based estimator is introduced along with the CLF, and its efficiency compared with a regular CLF strategy. Subsequently it is confronted with an identical control, and again, results are observed and studied. At last, the limitations encountered are studied and conclusions are obtained.

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Chapter 1

Introduction

In markets that operate under the hypothesis of pure competition, advertising always plays an important role. Through advertising, firms seek to call attention to their goods or services, attempting to induce consumers to prefer their products. This is done in order to maximize sales, and consequently, profits, which is the primary goal.

Given this fundamental role of advertising, it is no surprise that many studies have been made, seeking to model such dynamics. One of the earliest mathematical models of advertising was proposed by Dorfman and Steiner [1]. They assumed that a firm makes two kinds of choices: the price of its product and the amount of its advertising budget. They also postulated a functional relationship between the quantity the firm can sell per unit of time, q, its price, p, and its advertising budget, s, denoted as $q = f(p, s)$ and derived some basic qualitative results, using graphical methods and the tools of calculus. This pioneering work was soon followed by the influential paper of Vidale and Wolfe [2], who carried out detailed analysis of real firms and advertising data in order to propose their well known eponymous model, which is described as follows:

$$\dot{S} = rA(t)\frac{(M-S)}{M} - \lambda S \tag{1.1}$$

The model can be described by three parameters: λ, the exponential sales decay constant which models product obsolescence and reduces sales of a firm over time, M, the saturation level, which describes the maximum number of potential consumers, and r, the response constant, which represents the effectiveness of the advertising. This equation has the following interpretation: the increase in the rate of sales, $\dot{S}$, is proportional to the magnitude of the advertising effort, $A(t)$, reaching the fraction of potential customers, $\frac{(M-S)}{M}$, less the number of customers that are being lost due to obsolescence and forgetting, λS.

1.1 Duopoly in advertising models

The authors of the previously mentioned classical models performed their analysis using the term monopoly to denote the single firm models. Ozga [3] made one of the first studies of competitions and the diffusion of information and its interaction with the dynamics of advertising, effectively introducing the idea of competitive duopolistic models, which is the scenario used in this work. Another important class of models, closely related to the model used in this work, derives from the Lanchester model of human warfare [4], reprinted in [5], first applied to competition by Kimball [6], and reapplied to oligopolies by Little [7]. The latter model is described as follows:

$$\dot{s}_1 = \rho_1 x_1 s_2 - \rho_2 x_2 s_1 \tag{1.2}$$

$$\dot{s}_2 = \rho_2 x_2 s_1 - \rho_1 x_1 s_2 \tag{1.3}$$

where s_i is the sales rate of firm i, x_i is the rate of advertising of firm i and ρ_i is the advertising effectiveness constant of firm i.

Both the Vidale-Wolfe and Lanchester classes of models have been intensively studied theoretically and verified through experimental data (see, e.g., [8],[9], [10]).

Even though many of the early models were descriptive, they were conducive to the use of optimal control theory in order to derive the associated profit-maximizing dynamic advertising policies. In fact, a significant portion of the economics and management literature is focused on designing management strategies that optimize some predetermined performance indexes, using tools from optimal control (see [11], [12], [13], [14], [15], [16]).

Another approach is to recast the problem into a dynamic or differential game framework, to investigate possible Nash equilibrium that represent some possible equilibrium market shares [17], [18]. Finally, several studies have focused on the appearance of chaos in duopolistic or oligopolistic models (see, e.g., [19], [20] and references therein).

Some exceptions to these mainstream studies have also been published. The paper [21] studies different patterns of equilibrium without concentrating on the search for any optimal behavior. In [22], the authors exploit the different aspects of the dynamics of a duopolistic model also without a specific focus on optimality.

1.2 Models introducing churn

The term churn, sometimes also referred to as customer churn, refers to the loss of customers and can be thought of as the opposite of customer retention. It is a ubiquitous phenomenon that affects firms in almost all industries and therefore it is of interest to model the phenomenon adequately and provide a systematic analysis of the effects of churn. Specifically, for any given firm, since customers can both churn in and churn out (i.e., become or cease to be customers), it is necessary to analyze how the level of churn should affect advertising expenditure. One of the earliest attempts to model churn was in the classic Vidale-Wolfe model [2], which considered a "sales decay" term, which reduces the market share of a firm over time. This term was intended to model product obsolescence. In more recent times, according to Prasad and Sethi [23], there has occurred a "morphing of the sales decay term (...) into decay caused by competitive advertising and noncompetitive churn that acts to equalize market shares in the absence of advertising."

The churn effect intends to model product obsolescence, forgetting [2], lack of market differentiation [12], lack of information [13], variety seeking [14] and brand switching. These factors do not necessarily cause total market share to decay because the decay of market share for one firm is gain in market share for the other. Hence, we use the term churn rather than decay. Due to churn, the market shares converge to a long-run equilibrium when neither brand is advertised for a very long durations. The model used in this work takes into account decay due to competitive advertising as well as churn due to noncompetitive factors.

1.3 Objectives

The objective of this dissertation is to study the properties of the model with churn proposed in [24] in order to check its feasibility and adequacy to the real world. In order to achieve this we propose control strategies for the predatory advertising policy of a firm, in order for it to be able to achieve a desired market share. As a suggestion, we use a CLF based strategy. We also propose a way to estimate the competitor's predatory actions to improve the effectiveness of the CLF design. Finally, we discuss the results and the effectiveness of the proposed control technique, as well the consistency of the model with the real world, given the results attained with the CLF strategy.

1.4 Structure of dissertation

This dissertation is organized in five chapters. The first chapter presents a brief history of advertising models in duopoly markets and churn effects. Subsequently, a review of the linear churn model and the full churn model is presented. In the following chapter, a control Lyapunov function approach is proposed as a management policy. We also study the advantages and limitations of the technique, as well as the differences between a scenario in which a single firm uses the CLF approach, and a scenario in which both firms use the CLF approach. In the subsequent chapter, we propose an improvement on the technique, adding Levant's robust differentiator [25] to the CLF scheme to estimate the predatory actions that the firm's competitor is using. Finally, in the last chapter we review the main conclusions obtained in the dissertation, and suggest directions of future research.

Chapter 2

Models of advertising in duopolies with churn

In this chapter we will study a unified model of duopolistic dynamics that considers competition and churn, under the actions of different competitive advertising policies that is thought as different controls. This model was proposed in [24], and can be described as follows:

$$\dot{x}_1 = u_{11}s - u_{12}x_1 + u_{21}x_2 - k_1x_1 + k_2x_2 \quad (2.1)$$

$$\dot{x}_2 = u_{22}s + u_{12}x_1 - u_{21}x_2 + k_1x_1 - k_2x_2 \quad (2.2)$$

where

- x_1, x_2 are the state variables representing the market share fractions of firm 1 and firm 2 respectively (in the interval $[0,1]$),
- $s := 1 - x_1 - x_2$ is the unconquered market, also referred to as the untapped market or the market of undecided clients.
- u_{ii} is the action of firm i and represents positive advertising (i.e., advertising that increases the goodwill of the consumer and hence the market share x_i),
- the positive advertising control u_{ii} acts on s,
- $u_{ij}(x_1, x_2), i \neq j$ represents the predatory advertising (j on i),
- the predatory advertising control u_{ij} acts on x_i,
- the churn parameters k_i are non-negative numbers (in the interval $[0,1]$, if they are rates).

The unit for controls u_{ii} and u_{ij} are a measure of unit of money spent per unit of time. In this dissertation we will not use a specific unit such as dollars per month or dollars per year, since our main interest here is to study the dynamics of the system. All graphics presenting advertising efforts along this dissertation can be interpreted in the sense of any given unit of money spent per unit of time.

Furthermore, the controls presented here are always subject to some constraint in the real world. We consider that the money spent for advertising must be less or equal to the advertising budget of a firm. At first, we will not determine how much this maximum value must be. We will assume that the budget can support the amount spent.

In the following sections we will discuss the behavior of this model in detail, as well as the effects of some control policies on u_{ii} and u_{ij}.

2.1 Linear churn model

In this section the linear churn model is studied. The linear churn model can be considered a sub-model of equations (2.1) and (2.2), with additional constant churn parameters, and can be described as follows:

$$\dot{x}_1 = -k_{11}x_1 + k_{12}x_2 + \nu_1 \tag{2.3}$$

$$\dot{x}_2 = k_{21}x_1 - k_{22}x_2 + \nu_2 \tag{2.4}$$

The linear churn model can be used to study equilibrium points of the full churn model. If we consider that "due to churn, the market shares converge to a long-run equilibrium when neither brand is advertised for a very long duration" [23], we can assume that $u_{ii} = u_{ij} = 0, i, j, = 1, 2$. In this case, the equilibrium of the linear churn model determines the equilibrium of the full churn model in the absence of advertising.

This means that choice of parameters k_{ij} and ν_i is crucial in the determination of these equilibria. We state and prove a lemma in this regard.

Lemma 2.1. *The linear churn dynamics* (2.3)-(2.4) *possess a unique equilibrium if and only if* $\det K \neq 0$*, where*

$$K = \begin{bmatrix} -k_{11} & k_{12} \\ k_{21} & -k_{22} \end{bmatrix} \tag{2.5}$$

subject to the initial condition $(x_{10}, x_{20}) \in \mathbb{R}^2$*, with* $x_{i0} \geq 0, i = 1, 2$*. In this case, the unique equilibrium occurs at* $-K^{-1}\nu$*, where* $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$*. On the other hand, if* $\det K = 0$ *and* $\nu_1 = \nu_2 = 0$*, then the linear churn dynamics possesses infinitely many equilibria. In this case, without loss of generality, we can set* $k_{11} = k_{21} =: k_1$*,*

$k_{12} = k_{22} =: k_2$ *and, in this notation, all equilibria are located on the straight line through the origin with slope* k_1/k_2*. Specifically, trajectories emanating from the given initial condition* (x_{10}, x_{20}) *converge to the equilibrium* $(\frac{k_2}{k_1+k_2}(x_{10}+x_{20}), \frac{k_1}{k_1+k_2}(x_{10}+x_{20}))$ *located on this line.*

Proof: Equilibria must satisfy the equation $Kx = -\nu$, where $x = (x_1, x_2)$, from which the first statement for nonsingular K is obvious, in order to admit a nontrivial equilibrium. The more interesting case is for non-unique, nontrivial equilibria. Evidently, if $Kx = -\nu$, $Ky = -\nu$ then $K(x-y) = 0$, and for this homogeneous equation to admit a nontrivial solution, K must be singular. Singularity of the matrix K also means that its rows are multiples of each other and, in particular, we can choose the second row to be the negative of the first, justifying the choice $k_{11} = k_{21} =: k_1$, $k_{12} = k_{22} =: k_2$. From this viewpoint, this choice means that the so called churn-out, $-k_1x_1$, from firm 1 becomes the churn-in for firm 2 and vice-versa. With this choice, (i) all equilibria satisfy the equation $-k_1x_1 + k_2x_2 = 0$; (ii) isoclines of the trajectories are of slope $dx_2/dx_1 = -1$, proving the claim about the point $(\frac{k_2}{k_1+k_2}(x_{10}+x_{20}), \frac{k_1}{k_1+k_2}(x_{10}+x_{20}))$, which is the intersection of the isocline through (x_{10}, x_{20}) and the line $-k_1x_1 + k_2x_2 = 0$. □

Lemma 2.1 justifies the use of only two churn rate parameters k_i and $\nu_1 = \nu_2 = 0$ in the model (2.1)-(2.2), since we wish to avoid specifying the unknown long term equilibrium of the linear churn dynamics. Note, however, that the initial conditions (x_{10}, x_{20}) and the churn rate parameters k_1, k_2 do specify the long term equilibrium. These parameters can be and usually are estimated in real-life situations [26]. Observe also that the linear churn dynamics (2.3), (2.4) is the continuous-time equivalent of the discrete-time model proposed in [27].

Lemma 2.1 motivates the following terminology, used in the sequel. If $k_2 > k_1$, the equilibrium market share of firm 1 is larger than that of firm 2 and thus, in this case, we say that churn parameters are *favorable* to firm 1, and vice-versa when the inequality is reversed. Figure 2.1 exemplifies this by showing isoclines and the equilibrium line with slope $\frac{k_1}{k_2}$. This line divides the area of feasible market shares into two triangles and, since $k_2 > k_1$, the upper triangle is bigger than the lower one. As is possible to notice, every initial condition chosen inside the upper triangle is favorable for firm 1, since the red isoclines lines are pointing downwards and to the left, that is, the market share of firm 1 is increasing, while the market share of firm 2 is decreasing. An analogous interpretation holds for the lower triangle, with firm 2 having an advantage in this region. Even though there is a small region that favors firm 2, initially increasing its initial market share, it must be noticed that firm 1 always ends up with a greater market share than firm 2, that is, $x_{1f} > x_{2f}$.

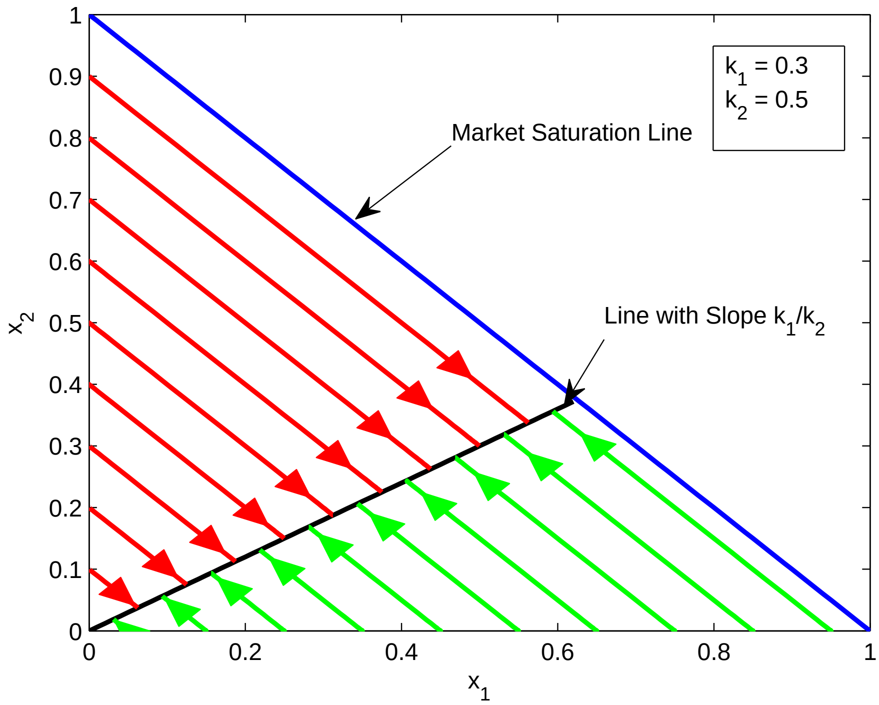

Figure 2.1: Phase plane for the linear churn model. The green lines are trajectories departing from initial conditions on the x_1 axis, while the red line are trajectories departing from initial conditions on the x_2 axis. In both cases, trajectories converge to equilibria on the line $x_2 = (\frac{k_1}{k_2})x_1$, as announced in lemma 2.1.

2.2 Full churn model

In this section we present and study the full churn model introduced in [24]. This model can be considered a generalization of the ideas contained in previous models presented in [23], [27], [28], and [29]. The model is rewritten here for convenience:

$$\dot{x}_1 = u_{11}s - u_{12}x_1 + u_{21}x_2 - k_1x_1 + k_2x_2 \tag{2.6}$$

$$\dot{x}_2 = u_{22}s + u_{12}x_1 - u_{21}x_2 + k_1x_1 - k_2x_2 \tag{2.7}$$

The justification of the particular choice of model (2.6),(2.7) is as follows. In an unsaturated market, $s > 0$, three types of terms are present in the dynamics, positive advertising acting on the unconquered market (s), predatory advertising acting on the competitor's market share and adding to the predator's share (observe that this obeys the principle of conservation of market share/clients) and, finally, churn, which also acts in such a way as to conserve total market share. When the market saturates, we have $x_1 + x_2 = 1$, which implies that $\dot{x}_1 + \dot{x}_2 = 0$ or, equivalently, $dx_2 = -dx_1$. In other words, if the market dynamics is to remain saturated ($s = 0$), the only way a firm can obtain more market share is from its competitor, conquering it by using predatory advertising, or by the churn effect. Moreover, we state that the equilibrium can only occur at a point on the line $x_1 + x_2 = 1$, as we show in Lemma 2.2.

Lemma 2.2. *The following assumptions are made on the advertising efforts u_{11}, u_{22}:*

A1. the sum of the advertising efforts $u_{11}(t) + u_{22}(t)$ is positive for all t;

A2. the sum of the advertising efforts is not L_1-integrable, i.e.:

$$\int_0^t (u_{11}(\tau) + u_{22}(\tau))\ d\tau \to \infty \text{ as } t \to \infty \tag{2.8}$$

Under assumptions A1 and A2, the dynamical system given by (2.6), (2.7) possesses the equilibrium set

$$S_{\text{eq}} := \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 + x_2 = 1\}. \tag{2.9}$$

Moreover, the set S_{eq} is globally attractive in the sense that all trajectories initiating in the positive quadrant $(x_1(0), x_2(0))$ tend exponentially to a point in S_{eq}.

Proof. At any equilibrium point (x_1^*, x_2^*), associated to $s^* = 1 - x_1^* - x_2^*$, of the dynamical system (2.6), (2.7), $\dot{x}_1 = \dot{x}_2 = 0$, i.e.:

$$u_{11}s^* - u_{12}x_1^* + u_{21}x_2^* - k_1x_1^* + k_2x_2^* = 0 \tag{2.10}$$

$$u_{22}s^* + u_{12}x_1^* - u_{21}x_2^* + k_1x_1^* - k_2x_2^* = 0 \tag{2.11}$$

Summing (2.10) and (2.11) yields:

$$(u_{11} + u_{22})s^* = 0 \tag{2.12}$$

By A1, (2.12) implies that $s^* = 0$. Conversely, if x_1^*, x_2^* are such that $s^* = 1 - x_1^* - x_2^* = 0$, then it is immediate that (2.10) and (2.11) are satisfied, proving that S_{eq} is the equilibrium set. To prove that S_{eq} is attractive, observe that $\dot{s} = -\dot{x}_1 - \dot{x}_2$, so that the ODE satisfied by s is

$$\dot{s} = -(u_{11}(t) + u_{22}(t))s \tag{2.13}$$

the solution of which is

$$s(t) = s(0)e^{\int_0^t -(u_{11}(\tau)+u_{22}(\tau))\ d\tau}$$

which shows, by A2, that $s(t) \to 0$ exponentially, for all initial conditions $s(0)$, as claimed. □

Remarks:

1. Note that Lemma 2.2 holds independently of the choice of u_{12}, u_{21} and does not specify which point in the equilibrium set S_{eq} is attained by a particular trajectory. In section 3.3, particular choices of the control efforts u_{12}, u_{21} are considered and the resulting equilibrium point in S_{eq} calculated.

2. Note that assumption A2 (and, of course, A1) hold if $u_{11}(t) + u_{22}(t)$ is a positive constant function. More generally, if $u_{11}(t) + u_{22}(t)$ tends to zero slower than $1/t$, then assumption A2 will be satisfied.

In summary, the construction of this model is based on the following assumptions:

A1 In the unsaturated market, positive advertising by firm i acts only on the unconquered market.

A2 Negative or predatory advertising occurs only when firm i has a market share larger than that of firm j and, in this case, acts with constant effort by firm j on the share of firm i, i.e., through a term $-c_{ij}x_i$.

A3 The market share lost by firm i due to this predatory action is gained by firm j.

A4 In the saturated state, only predatory advertising is present.

A5 Churn is always present, in both saturated and unsaturated markets.

Assumption A1 is a standard one in the literature, dating from the seminal paper [2]. Assumption A2 is a simple version of optimizing behavior: a firm only begins to invest in predatory advertising when it realizes that it has a lower market share than its competitor. Assumption A3 reflects the belief that, in a duopoly for an essential good, the market share lost by one firm is necessarily gained by the other, since clients need the good, no matter which firm is supplying it, and therefore cannot simply disappear. Assumption A4 has the following natural interpretation. Once the market is saturated, there are no more undecided clients, thus any attempt by either firm to increase its market share must necessarily be predatory, reducing its competitor's share. Assumption A5 is natural because churn is essentially a client driven phenomenon and therefore present independently of the control strategies used by the firms.

After presenting this model and its inherent properties, we develop a control technique to try to achieve a desired market share. A variety of switching policies combining linear and constant controls were proposed in [24] for positive and predatory advertising, in order to study these possibilities. This work proposes a more complex control, by using Control Lyapunov Functions, from now on denoted as CLF. We will make use of constant positive advertising, while we will develop a CLF scheme for predatory advertising.

The use of an estimator of the competitor's predatory effort is also proposed, in order to analyze how it might improve the control effectiveness.

After constructing these new techniques, we will be able to verify the consistency of the model and to understand the model's dynamic more deeply. In order to systematize our study, the following questions will be raised and answered, in the chapters that follow:

- Q1 Is it possible for a single firm using the CLF control, to achieve a desired market share?
- Q2 Is this still possible, if the firm does not know, precisely, how much advertising effort its competitor is using?
- Q3 What happens if both firms use CLF control to achieve their desired market share in the case when the desired shares add up to more than 1?
- Q4 Does the use of an estimator (of its competitor's actions/parameters) by one firm confer some advantage to it, assuming that its competitor does not use an estimator?
- Q5 What happens when both firms use the same technique with the same information?

Chapter 3

Control design using CLF method

In this chapter a control law using CLF method will be developed for the system described by equations (3.1) and (3.2), repeated here for convenience:

$$\dot{x}_1 = u_{11}s - u_{12}x_1 + u_{21}x_2 - k_1x_1 + k_2x_2 \tag{3.1}$$

$$\dot{x}_2 = u_{22}s + u_{12}x_1 - u_{21}x_2 + k_1x_1 - k_2x_2 \tag{3.2}$$

The proposed control law will be applied and studied in different scenarios. In the next section, we start with a brief review of the CLF method.

3.1 CLF Method

In this section we present the Control Lyapunov Function method briefly. First we consider an autonomous system:

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{3.3}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. We also suppose that $\mathbf{x} = \mathbf{0}$ is an equilibrium point. In order to $\mathbf{x} = \mathbf{0}$ to be stable or asymptotically stable, the following theorem must hold:

Theorem 3.1. *Let $\boldsymbol{x} = \boldsymbol{0}$ be an equilibrium point for* (3.3) *and $D \subset \mathbb{R}^n$ be a domain containing $\boldsymbol{x} = \boldsymbol{0}$. Let $V : D \to \mathbb{R}$ be a continuous differentiable function such that:*

$$V(\boldsymbol{0}) = 0 \ \ \textit{and} \ \ V(\boldsymbol{x}) > 0 \textit{ in } D - \{\boldsymbol{0}\} \tag{3.4}$$

$$\dot{V}(\boldsymbol{x}) \leq 0 \textit{ in } D \tag{3.5}$$

Then, $\boldsymbol{x} = \boldsymbol{0}$ *is stable. Moreover, if*

$$\dot{V}(\boldsymbol{x}) < 0 \ in \ D - \{\boldsymbol{0}\} \tag{3.6}$$

then $\boldsymbol{x} = \boldsymbol{0}$ *is asymptotically stable.*

A function $V(\mathbf{x})$ satisfying (3.4) and (3.5) is called a Lyapunov Function. A proof of Theorem 3.1 can be found in [30]. It must be noticed that Theorem 3.1 also holds for a non zero equilibrium point, that is $\mathbf{x} \neq \mathbf{0}$. This can be achieved by making the transformation $\mathbf{z} = \mathbf{x} - \mathbf{x}_e$, where $\mathbf{x}_e$ is the non-zero equilibrium. This transformation generates an autonomous system equivalent to (3.3), and $\mathbf{x} = \mathbf{x}_e$, implies that $\mathbf{z} = \mathbf{0}$, which is sufficient, along with an appropriate function $V(\mathbf{z})$, to create the same premises necessary for Theorem 3.1 to hold.

The Control Lyapunov Function method uses Theorem 3.1 to guarantee asymptotic stability for a desired equilibrium point of a given system by chosing an appropriate input function $u(t)$. Let us suppose that we have now the following system:

$$\dot{\mathbf{x}} = f(\mathbf{x}, u) \tag{3.7}$$

The key idea here is to choose a Lyapunov Function $V(x)$ where $u(\mathbf{x})$ is constructed in such a way that (3.4) and (3.6) hold. Of course, u must be a function of $\mathbf{x}$ only in order that (3.7) continue to be an autonomous system. Designing $V(\mathbf{x})$ and u this way will make it possible to apply Theorem (3.1).

In the next section we will design both $V(\mathbf{x})$ and u for the system described by equations (3.1) and (3.2).

3.2 Control applied by a single firm

In this section a CLF will be proposed for the system described by equations (3.1) and (3.2), assuming that it is desired to attain the equilibrium point, $\mathbf{x}_d = (m_1, 1 - m_1)$. We start by defining the Lyapunov Function, $V(\mathbf{x})$.

$$V(\mathbf{x}) = \frac{1}{2}(x_1 - m_1)^2 + \frac{1}{2}(x_2 - (1 - m_1))^2 \tag{3.8}$$

The value $\mathbf{x}_d = (m_1, 1 - m_1)$ is the only point that makes $V(\mathbf{x}) = 0$ and corresponds to the market share scenario desired by firm 1. These conditions on $V(\mathbf{x})$ are the first step towards making $\mathbf{x}_d$ the unique attractor of the system. Differentiating the expression for $V(x)$ with respect to time along the trajectories of (3.1) and (3.2) leads to equation (3.9). Also, we make the following assumption:

Assumption 3.2. *The advertising efforts $u_{11}(t)$, $u_{22}(t)$ are assumed to be constants denoted by c_{11}, c_{22} respectively.*

We will choose c_{11} and c_{22} according to each situation, rather than fixing predetermined values. Also, these values will be changed within an analysis in order to verify its influence in the model.
After these considerations, then $\dot{V}(\mathbf{x})$ will be as follows:

$$\dot{V}(\mathbf{x}) = [(x_1 - m_1)c_{11} + (x_2 - 1 + m_1)c_{22}](1 - x_1 - x_2) + a[(u_{21} + k_2)x_2 - (u_{12} + k_1)x_1] \quad (3.9)$$

where $a = x_1 - x_2 - 2m_1 + 1$.

In order to obtain a stable closed loop controlled system, we need to ensure that $\dot{V}(\mathbf{x})$ is negative definite, and to this end we will use the yet to be defined parameter u_{21}. We will make $\dot{V}(\mathbf{x})$ negative by the particular choice of the control u_{21} that makes $\dot{V}(\mathbf{x}) = -2V(\mathbf{x})$, since this implies that V and hence $\mathbf{x}(t)$ (roughly) decay like e^{-2t}. Thus:

$$\dot{V}_d(\mathbf{x}) = -(x_1 - m_1)^2 - (x_2 - (1 - m_1))^2 \quad (3.10)$$

Also, notice that $\mathbf{x}_d = (m_1, 1 - m_1)$ is the only point that makes $\dot{V}(\mathbf{x}_d) = 0$. This last property fully characterizes $\mathbf{x}_d$ as the unique equilibrium. Equating (3.9) and (3.10), the following expression for u_{21} is obtained after algebraic manipulations:

$$u_{21} = \frac{u_{21}^{\dagger} + (u_{12} + k_1)x_1}{x_2} - k_2 \quad (3.11)$$

where

$$u_{21}^{\dagger} = \frac{-[(x_1 - m_1)c_{11} + (x_2 - 1 + m_1)c_{22}]s - (x_1 - m_1)^2 - (s + x_1 - m_1)^2}{s + 2(x_1 - m_1)} \quad (3.12)$$

Initially we will not be concerned about the positivity of u_{21}, even though negative or unbounded values of the control variable are not feasible in the theoretical model. These restrictions will be considered later on in section 3.4.

We assume now that only firm 1 uses control designed via a CLF, while firm 2 uses a constant effort control. We also assume that firm 2 does not use a CLF strategy, and maintains a constant effort to predate firm's 1 market share, which will be denoted as $u_{12} = c_{12}$. This value will be chosen according to each situation, in order to achieve a better understanding of the system dynamics.

Furthermore, firm 1 is considered to not have any previous knowledge of constant c_{12}, nor of c_{22}, which represents the action of firm 2 on the unconquered market. Thus, estimates $\hat{c}_{12}$ and $\hat{c}_{22}$ replace these constants in expressions (3.11) and (3.12) for u_{21} and $u_{21}^{\dagger}$ respectively. All other constant parameters are assumed to be known, and market shares x_1 and x_2 are assumed known (in real time).

Figure 3.1 shows the trajectory that the system follows for a initial condition $\mathbf{x}_0 = (0.2, 0.3)$ and final objective $m_1 = 0.8$. The churn parameters k_i and positive advertising parameters c_{ii} were chosen in order to create an unfavorable scenario for firm 1.

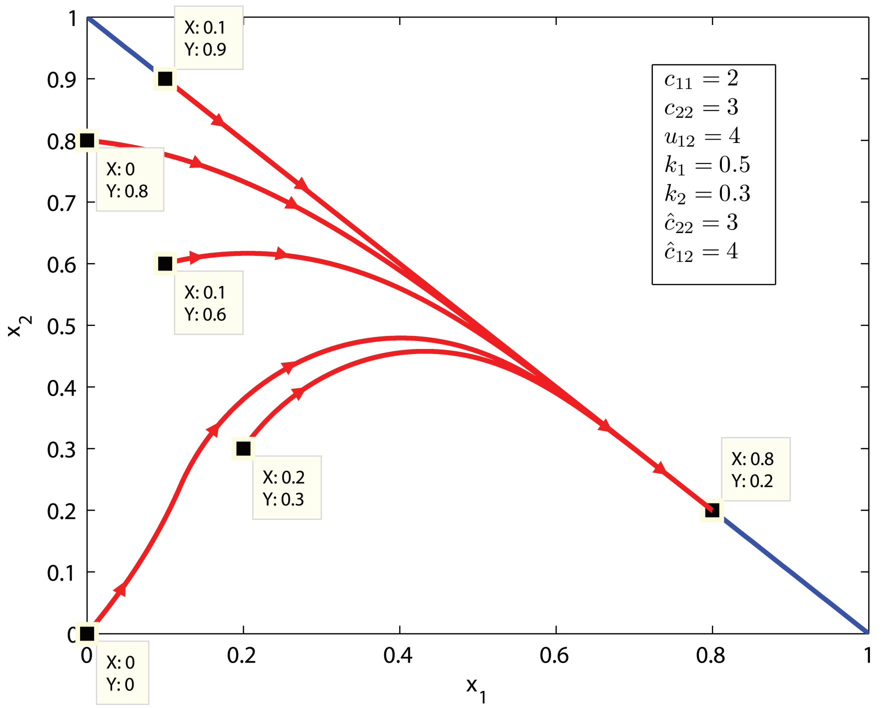

Figure 3.1: Trajectory for a set of different initial conditions and $m_1 = 0.8$. The technique is successful and firm 1 achieves the desired market share.

As we can see, the technique is successful, and makes it possible for firm 1 to achieve the desired market share in finite time, despite an unfavorable initial condition, a smaller value of positive advertising, as well as an adverse choice of churn parameters. Another important feature of this design is the fact that the control signal is bounded and feasible, as we can see in figure 3.2

After these considerations, we can state the following theorem:

Theorem 3.3. *If firm 2 uses constant predatory advertising, and both firms use constant positive advertising, then firm 1 can achieve any desired market share from any initial market share, using a CLF predatory control with a precise estimate of predatory control of firm 2. This result only holds if no restrictions are imposed on the magnitude of predatory control of firm 1.*

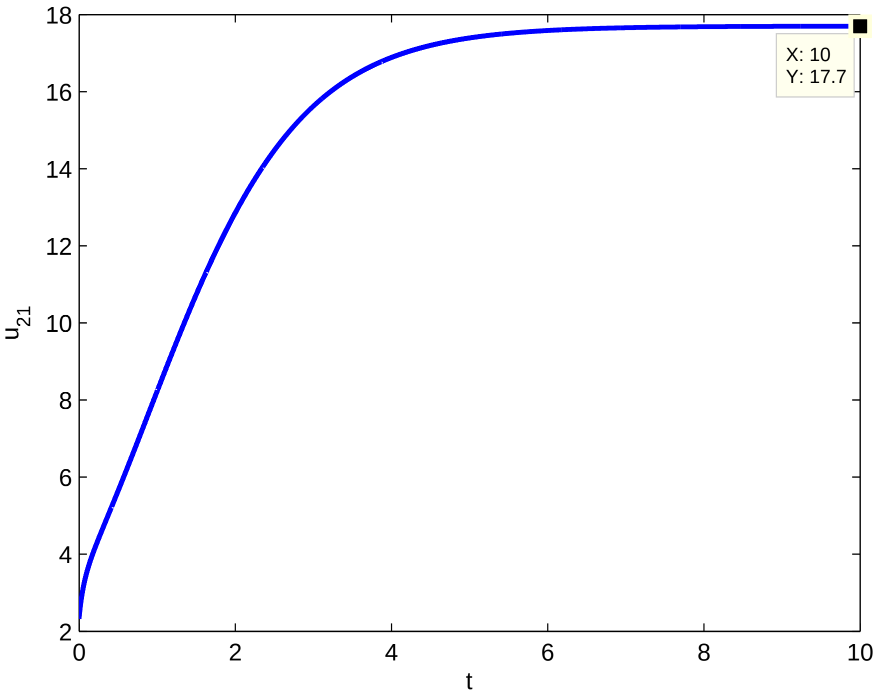

Figure 3.2: Control Signal for firm 1 for the case with initial conditions $\mathbf{x}_0 = (0.2, 0.3)$. Represents the amount of predatory advertising firm 1 uses. As mentioned before, we will not use a specific unit since we are interested in studying the dynamics, and therefore this graphic must be understood as amount of money spend during a certain period of time

Proof: In order to find the equilibrium points of the controlled system, we apply (3.11) in (3.1) and (3.2) and equate them to zero, yielding the following linear system of algebraic equations:

$$c_{11}(1 - x_1 - x_2) + u^\dagger = 0 \tag{3.13}$$

$$c_{22}(1 - x_1 - x_2) - u^\dagger = 0 \tag{3.14}$$

The unique solution of (3.13) and (3.14) is the point $(m_1, 1 - m_1)$ is the unique solution. The asymptotic stability of this point within the domain of interest is guaranteed by the CLF design of $V(\mathbf{x})$.

3.3 Sensitivity of CLF design to errors in estimates

This subsection investigates the sensitivity of the control design with respect to errors in the estimates $\hat{c}_{12}$ and $\hat{c}_{22}$, since so far they have been considered to be perfectly accurate. For the subsequent analysis, all the other parameters are considered to be identical to the previous case, as shown in figure 3.1. Simulations in figure 3.3 indicate that parameter $\hat{c}_{22}$ has only a small effect on the transient response, making it slightly slower for underestimated values. The effect on the state trajectory is also presented in figure 3.4. In regard to the control signal, figure 3.5 indicates that the more we underestimate the value of $\hat{c}_{22}$, the smaller the L_2 norm of the control signal will be. Conversely, the more we overestimate it, the bigger this norm of the control signal will be. Finally, from figures 3.3 and 3.4, it can also be observed that the target market share is attained by firm 1 in all cases. After these experimental observations, it is reasonable to make the following assumption:

Assumption 3.4. *Consider two different estimates for parameter c_{ii}, $\hat{c}_{ii1}$ and $\hat{c}_{ii2}$. If $\hat{c}_{ii1} > c_{ii2}$ then $\|u_1\|_2 > \|u_2\|_2$, where $\|.\|_2$ represents the L_2 norm of a function, and u_1 and u_2 are the control signals generated with the respective estimates.*

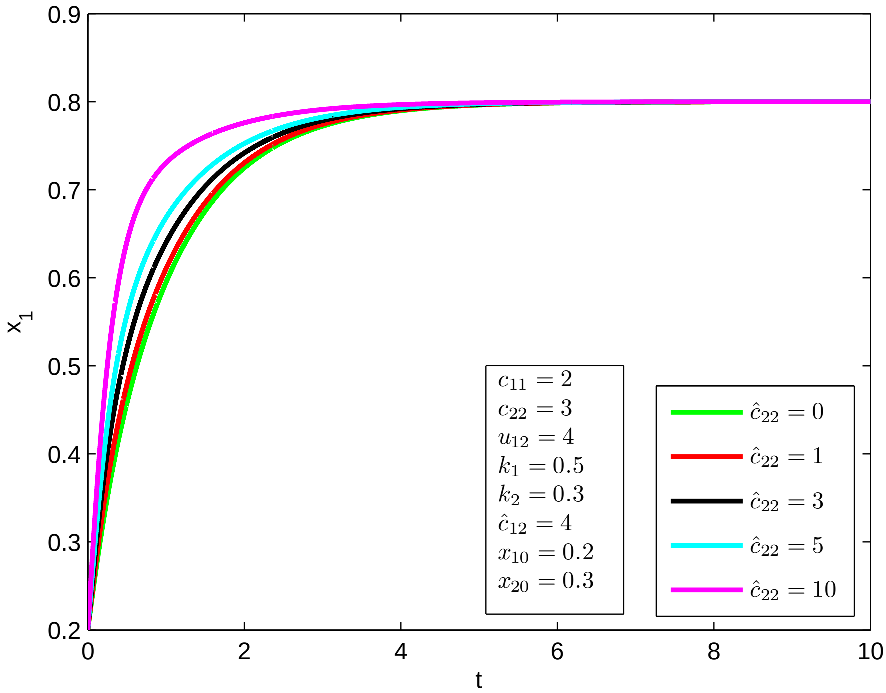

Figure 3.3: The trajectories of x_1 do not differ much, as parameter $\hat{c}_{22}$ is varied, and all others are kept fixed, showing the low sensitivity of x_1 with respect to the parameter $\hat{c}_{22}$.

On the other hand, when the estimate $\hat{c}_{12}$ deviates from its correct value, more severe changes occur. The most notable one is that the targeted market share may not be reached. If $\hat{c}_{12}$ is underestimated, the steady state market share of firm 1, x_{1f}, will be lower than the intended market share m_1. Similarly, if $\hat{c}_{12}$ is overestimated we get $x_{1f} < m_1$. From this observation it is possible to enunciate the following theorem:

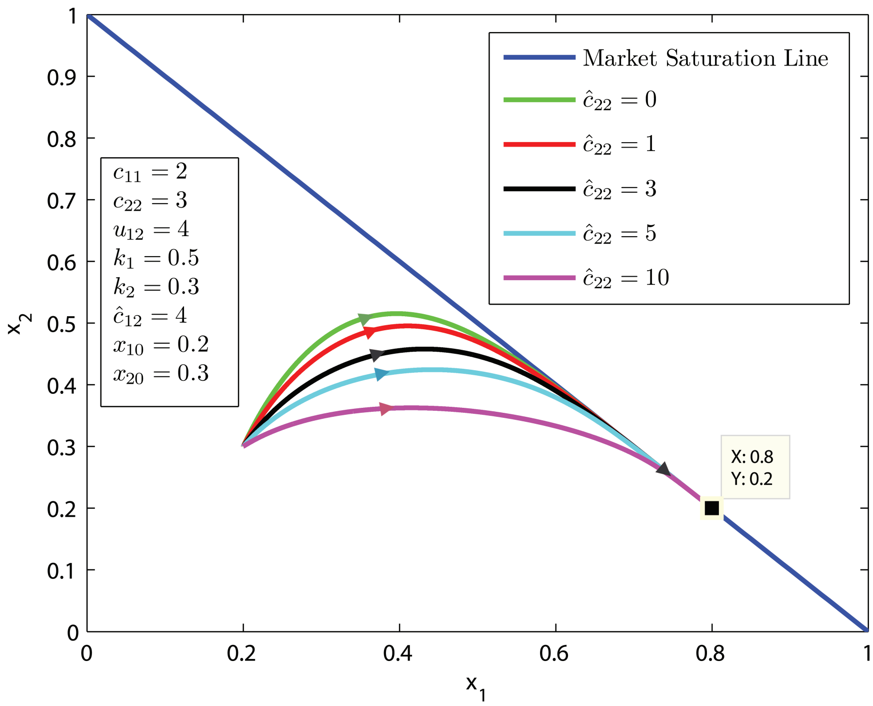

Figure 3.4: Sensitivity of trajectory with respect to the parameter $\hat{c}_{22}$. Notice that the final state does not change.

Theorem 3.5. *[Calculation of specific equilibrium point in S_{eq} resulting from the application of CLF control, under the assumption of estimation error in $\hat{c}_{12}$] Consider the dynamical system* (3.1), (3.2) *subject to the CLF-designed control u_{21} given by* (3.11) *and the control $u_{12} = c_{12}$ (constant). The equilibrium point $\boldsymbol{x}_f \in S_{\text{eq}}$ attained under these circumstances is given by:*

$$\boldsymbol{x}_f = \left(\frac{u_{21f} + k_2}{c_{12} + u_{21f} + k_1 + k_2}, \frac{c_{12} + k_1}{c_{12} + u_{21f} + k_1 + k_2} \right) \tag{3.15}$$

If c_{12} is overestimated, i.e. $\hat{c}_{12} > c_{12}$ then $x_{1f} > m_1$. Conversely, if c_{12} is underestimated, i.e. $\hat{c}_{12} < c_{12}$ then $x_{1f} < m_1$.

Proof: First we recall that the result of Lemma 2.2 holds for all choices of u_{12}, u_{21}, including those of the statement of this theorem and it remains to solve the equilibrium equations for these specific choices. In other words, we wish to solve:

$$u_{11}(1 - x_{1f} - x_{2f}) - u_{12}x_{1f} + u_{21}x_{2f} - k_1x_{1f} + k_2x_{2f} = 0 \tag{3.16}$$

$$u_{22}(1 - x_{1f} - x_{2f}) + u_{12}x_{1f} - u_{21}x_{2f} + k_1x_{1f} - k_2x_{2f} = 0 \tag{3.17}$$

From Lemma 2.2, we may put $s = (1 - x_1 - x_2) = 0$, so that (3.16), (3.17) simplify

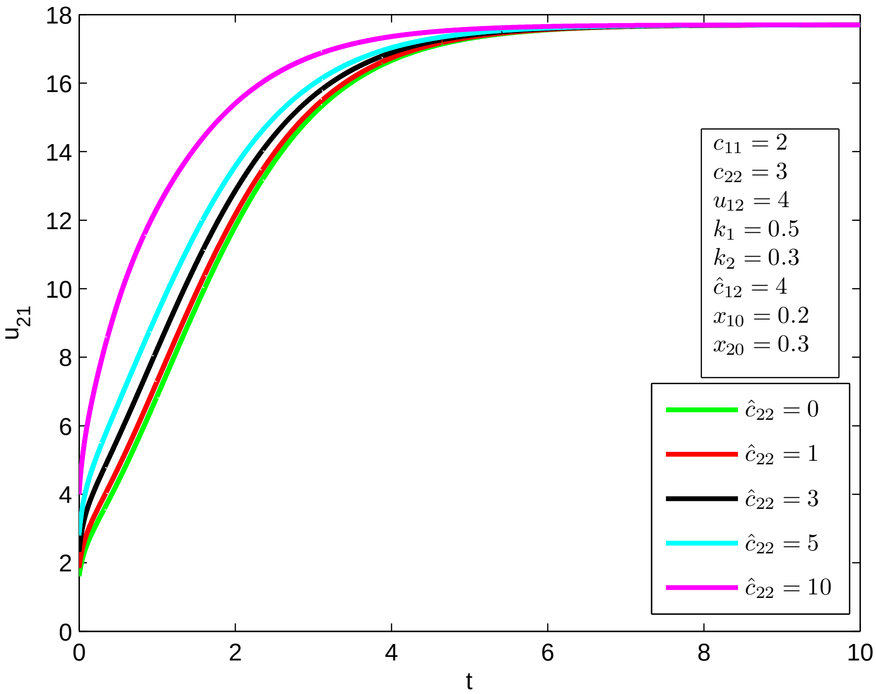

Figure 3.5: Sensitivity of control variable with respect to the parameter $\hat{c}_{22}$. Error in the estimate results in larger predatory advertising effort spent if overestimated, and smaller predatory advertising effort spent if underestimated.

to:

$$-u_{12}x_{1f} + u_{21f}x_{2f} - k_1x_{1f} + k_2x_{2f} = 0 \tag{3.18}$$

$$u_{12}x_{1f} - u_{21f}x_{2f} + k_1x_{1f} - k_2x_{2f} = 0 \tag{3.19}$$

Equations (3.18) and (3.19) are linearly dependent so it is only possible to solve for x_{1f} in terms of x_{2f}. Substitution of $u_{12} = c_{12}$ and, from (3.11), $u_{21f} = \frac{u^{\dagger}_{21f} + (\hat{c}_{12} + k_1)x_{1f}}{x_{2f}} - k_2$ yields:

$$-c_{12}x_{1f} + \hat{c}_{12}x_{1f} + u^{\dagger}_{21f} = 0 \tag{3.20}$$

From (3.12), $u^{\dagger}_{21f} = -(x_{1f} - m_1)$, since $x_{2f} = 1 - x_{1f}$, which implies that

$$(\hat{c}_{12} - c_{12})\, x_{1f} = x_{1f} - m_1 \tag{3.21}$$

Since $x_{1f} > 0$, (3.21) implies the statements about under- and overestimation of the constant control c_{12}.

From (3.18) and the fact that $x_{1f} + x_{2f} = 1$, it follows directly that the solution of the linear system (3.18), (3.19) is given by (3.15), as claimed. □

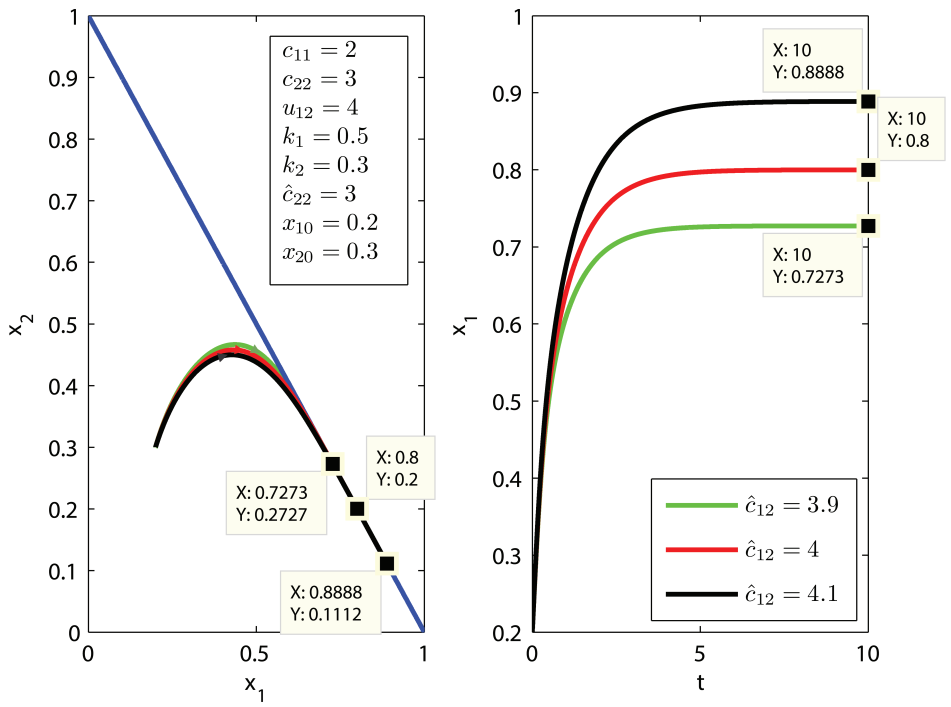

Figure 3.6: Sensitivity of trajectory with respect to the parameter estimate $\hat{c}_{12}$. Over estimates and under estimates produce corresponding errors in the final state.(below (respectively above) desired value of $m_1 = 0.8$)

Figure 3.6 illustrates state trajectories that result when the inequalities presented in theorem 3.5 are imposed. Control signals are also shown in figure 3.7. These results are intuitively clear. If firm 2's predation effort is actually bigger than what was predicted for it, u_{21} is designed in such a way that it has a smaller magnitude than what it should have had. The immediate consequence is that firm's 1 market share ends up being smaller than what was planned. If the estimate is far below the real value of c_{12}, the results can be disastrous, with firm 1 having almost no market share at the end.

Similarly, if the estimated predation of firm 2 is smaller than what occurs in reality, u_{21} has a greater magnitude than required, and firm 1's market share ends up being greater than what was planned. This result could also be undesirable since the extra market share obtained may not compensate the extra advertising costs, leading to a loss in the long term.

This can be enunciated analytically, manipulating equation (3.21) to obtain a relation between percentage error of $\hat{c}_{21}$ and x_{1f}

$$\epsilon_{x_f} = \frac{\epsilon_c}{\frac{1}{c_{21}} - \frac{\epsilon_c}{100}} \tag{3.22}$$

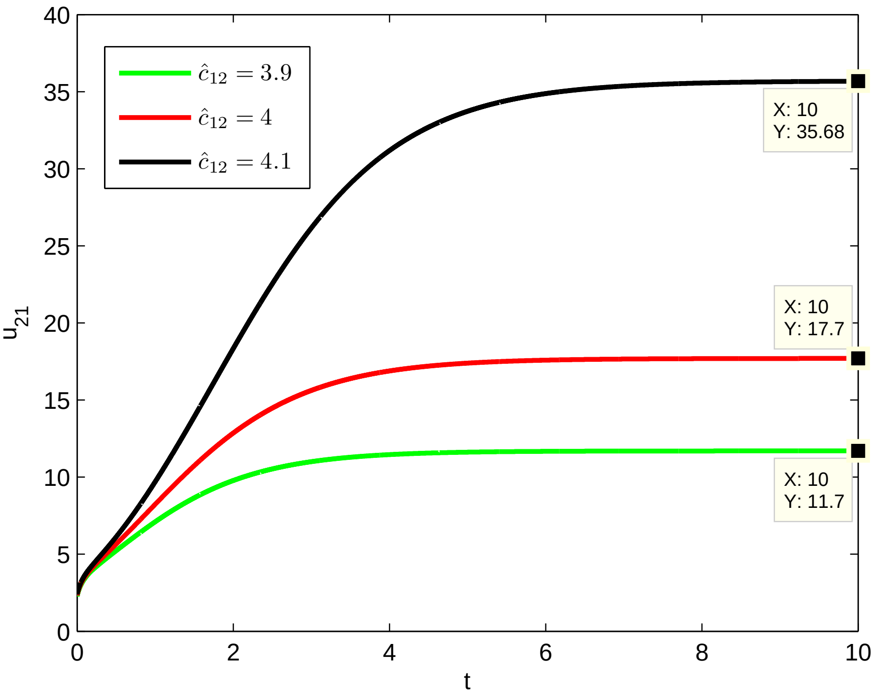

Figure 3.7: Sensitivity of control variable with respect to the parameter $\hat{c}_{12}$. Notice that small deviations in the value of $\hat{c}_{12}$ produces a big change in the control variable, exposing a high sensitivity to this parameter.

where $\epsilon_{x_f} = \frac{100(x_{1f}-m_1)}{m_1}$ and $\epsilon_c = \frac{100(\hat{c}_{21}-c_{21})}{c_{21}}$. The graph showing the relation of both percentage errors is shown in figure 3.8.

The main conclusion that can be drawn from this section is that the proposed control technique works fine as long as we have a good estimate for c_{12} based on equation (3.22). This task can be quite difficult since small errors in estimate $\hat{c}_{21}$ can produce a considerable error in the final state x_{1f}. Another question that arises immediately is the following: what happens if firm 2 also uses a similar CLF scheme? This is addressed in the next section. We return to the question of estimation of parameters in chapter 4.

3.4 CLF designed control applied by both firms

In this section, we study the effects of both firms using the CLF method in order to achieve a certain market share. Initially, an expression for u_{12} is required. This can be done in a very similar way as was previously done for u_{21}. The first step is to define a new CLF $V(\mathbf{x})$ as follows:

$$V(\mathbf{x}) = \frac{1}{2}(x_1 - (1 - m_2))^2 + \frac{1}{2}(x_2 - m_2)^2 \tag{3.23}$$

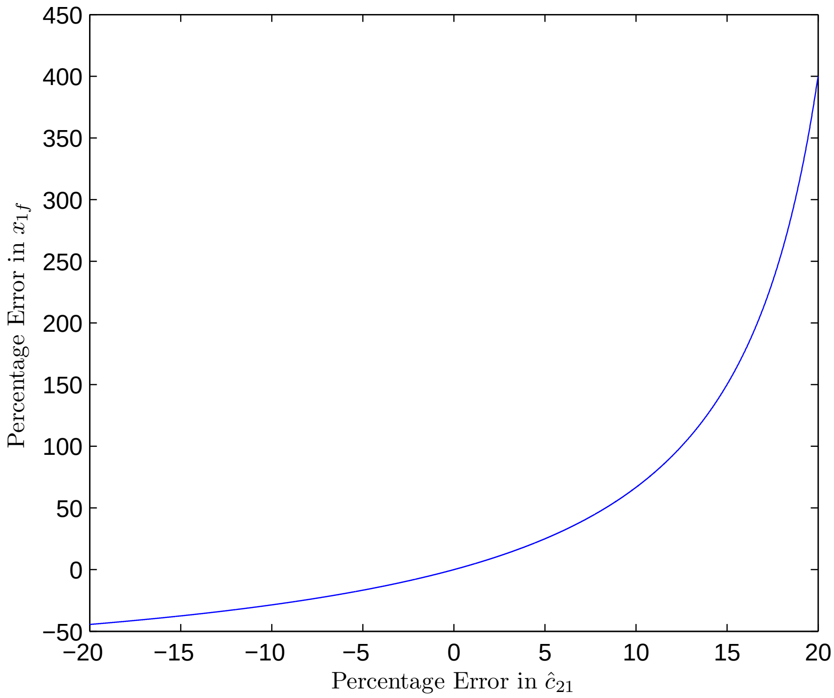

Figure 3.8: Percentage error of the estimate $\hat{c}_{21}$ versus percentage error of the final market share x_{1f} with respect to the desired market share m_1, for $m_1 = 0.8$ and $c_{21} = 4$. This plot is only illustrative, since errors that produce $x_{1f} > 1$ do not have any practical meaning.

Then,

$$\dot{V}_d(\mathbf{x}) = -(x_1 - (1 - m_2))^2 - (x_2 - m_2)^2 \tag{3.24}$$

Following steps similar to those in section 3.2, one finds:

$$\dot{V}(\mathbf{x}) = [(x_2 - m_2)c_{22} + (x_1 - 1 + m_2)c_{11}]s + b[(u_{12} + k_1)x_1 - (u_{21} + k_2)x_2] \tag{3.25}$$

where $b = x_2 - x_1 - 2m_2 + 1$. And therefore:

$$u_{12} = \frac{u_{12}^{\dagger} + (u_{21} + k_2)x_2}{x_1} - k_1 \tag{3.26}$$

where

$$u_{12}^{\dagger} = \frac{-[(x_1 - 1 + m_2)c_{11} + (x_2 - m_2)c_{22}]s - (s + x_2 - m_2))^2 - (x_2 - m_2)^2}{s + 2(x_2 - m_2)} \tag{3.27}$$

Again, as in the single firm case, firm 2 does not have any knowledge about constant c_{11} and control signal u_{21}, and they were replaced by constant estimates $\hat{c}_{11}$ and $\hat{c}_{21}$ in equations (3.26) and (3.27).

It should be noticed that the inclusion of the objective market share of firm 2 may lead to a situation in which $m_1 + m_2 > 1$. We know *a priori* that this situation is infeasible. It is natural to assume that the firms will struggle for their objectives with all the predatory effort that is available to them.

So far we have not raised any concerns about the boundedness of the control signals, since they were limited by some value and we were assuming that the budget was greater than it. But in this situation we have two continuous control functions, struggling with opposing forces for more market share. It may therefore be expected that one, or both could increase their magnitude arbitrarily until they get what they planned. In this case we need to define how much a firm is able to spend with advertising. Furthermore, negative numbers could result in equations (3.11) and (3.26). It thus becomes necessary to impose some physical limit on the controls u_{ij}, since it is not feasible to have a firm spending negative money on advertising, or spending an arbitrarily large amount of money on this activity. For further analysis, we will assume that:

$$0 \leq u_{ij}(t) \leq u_{\max},$$

for all $t > 0$.

The value of $u_{\max}$ will be interpreted as a firm using its entire budget for predation, equal to 100 for the sake of exemplification. This value can be changed without any loss of generality.

The behavior of the system was simulated subject to these assumptions. It is possible to check in figure 3.9 that the controls are identical, since both firms are under equal initial conditions, and have the same goals and parameters. This is an expected outcome, since both controls were designed in the same way. If no firm were given an advantage, they should indeed end up on the 50-50 market share point as they are seen to do. Control signals u_{12} and u_{21} are also shown in figure 3.10.

The next obvious step is to check what happens when one or more parameters change, giving an edge to a firm. In figure 3.11, the effects on state trajectory of changing initial condition x_{10}, *ceteris paribus*, are shown.

The outcome is an outright victory of a firm depending on who has the bigger initial condition. When $x_{10} = 0.15$ firm 1 conquered almost the entire market share, saturating its control variable (See figure 3.12). But if a similar experiment is made with a different choice of parameters, the outcome is very different as can be seen in figure 3.13. The control variables when $x_{10} = 0.15$ are also shown in figure 3.14.

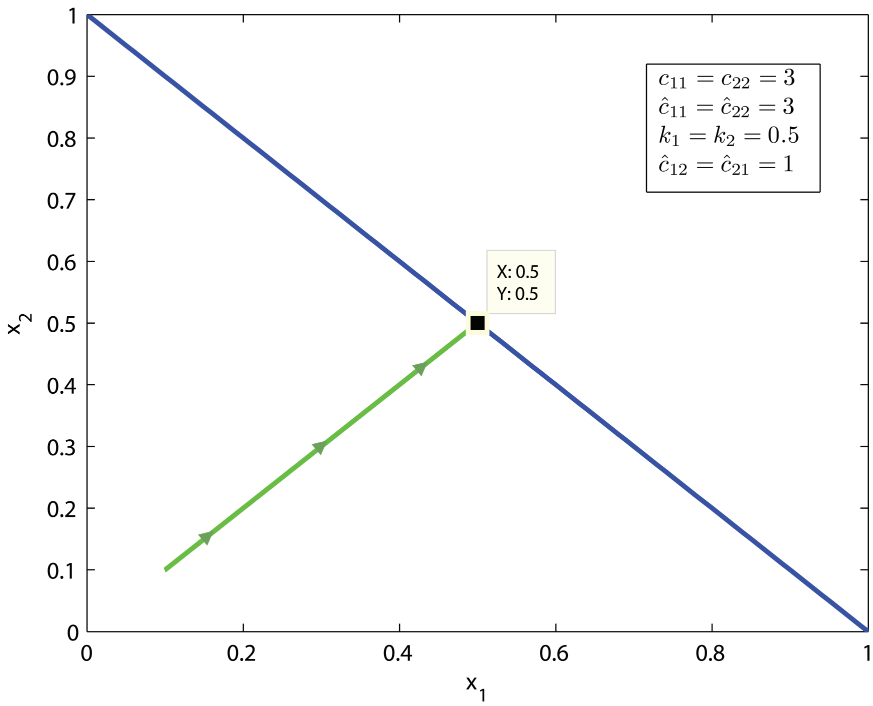

Figure 3.9: State trajectory for $m_1 = m_2 = 0.9$ and $x_{10} = x_{20} = 0.1$. Notice that under identical conditions, the use of the technique by both firms leads them to an equal division of the market.

It is clear that further investigation of the stability of equilibrium points is needed to better interpret these graphs. In order to do that, the values of u_{12} and u_{21} (Equations (3.26) and (3.11)) will be applied in the dynamic equations (Equations (3.1) and (3.2)). Also, we know from Lemma 2.2 that the line $s = 0$ is the only locus that can contain equilibrium points. Substituting accordingly, this leads to the following expressions:

$$\dot{x}_1 = -(x_1 - m_1) + (x_2 - m_2) + k_1 x_1 - k_2 x_2 + \hat{c}_{12} x_1 - \hat{c}_{21} x_2 \tag{3.28}$$

$$\dot{x}_2 = (x_1 - m_1) - (x_2 - m_2) - k_1 x_1 + k_2 x_2 - \hat{c}_{12} x_1 + \hat{c}_{21} x_2 \tag{3.29}$$

Equating the right hand side of these equations to zero and recalling that $x_1 + x_2 = 1$, allow us to calculate the following unique equilibrium point:

$$\mathbf{x}_e = \left(\frac{k_2 + \hat{c}_{21} - 1 + m_2 - m_1}{k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2}, \frac{k_1 + \hat{c}_{12} - 1 + m_1 - m_2}{k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2} \right) \tag{3.30}$$

If we analyze the previous case (Figure 3.9 and figure 3.11), where all parameters were identical, the equilibrium point will be $(0.5, 0.5)$. It remains to examine the stability of this point. This is done in theorem 3.6:

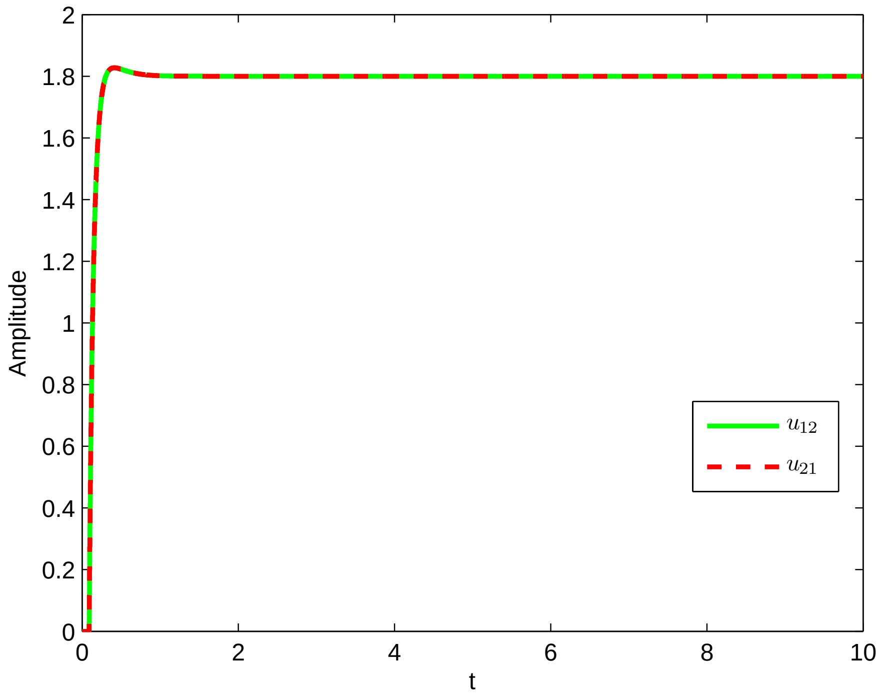

Figure 3.10: Control signals for a scenario in which both firms use CLF and have the same goals and the same amount of informations from its competitor

Theorem 3.6. *If both firms use a CLF predatory control, then the equilibrium point of the system is a stable node if* $k_1+k_2+\hat{c}_{12}+\hat{c}_{12}-2<0$. *Otherwise, if* $k_1+k_2+\hat{c}_{12}+\hat{c}_{12}-2>0$, *then the equilibrium point is a saddle*

Proof for theorem 3.6 is as shown:

In order to demonstrate Theorem 3.6, first we substitute the predatory controls in the duopoly system, represented in (2.1) and (2.2) with proposed controls (3.11) and (3.26).

$$\dot{x}_1 = c_{11}\left(1 - x_1 - x_2\right) - u_{12}^{\dagger} - \hat{c}_{21}x_2 + u_{21}^{\dagger} + \hat{c}_{12}x_1 + k_1x_1 - k_2x_2 \tag{3.31}$$

$$\dot{x}_2 = c_{22}\left(1 - x_1 - x_2\right) + u_{12}^{\dagger} + \hat{c}_{21}x_2 - u_{21}^{\dagger} - \hat{c}_{12}x_1 - k_1x_1 + k_2x_2 \tag{3.32}$$

We now evaluate the Jacobian of system (3.31) and (3.32):

$$J = \begin{bmatrix} -c_{11} - \frac{\partial u_{12}^{\dagger}}{\partial x_1} + \frac{\partial u_{21}^{\dagger}}{\partial x_1} + k_1 + \hat{c}_{12} & -c_{11} - \frac{\partial u_{12}^{\dagger}}{\partial x_2} + \frac{\partial u_{21}^{\dagger}}{\partial x_2} - k_2 - \hat{c}_{21} \\ -c_{22} + \frac{\partial u_{12}^{\dagger}}{\partial x_1} - \frac{\partial u_{21}^{\dagger}}{\partial x_1} - k_1 - \hat{c}_{12} & -c_{22} + \frac{\partial u_{12}^{\dagger}}{\partial x_2} - \frac{\partial u_{21}^{\dagger}}{\partial x_2} + k_2 + \hat{c}_{21} \end{bmatrix} \tag{3.33}$$

Evaluating the partial derivatives of controls u_{12} and u_{21}, one obtain:

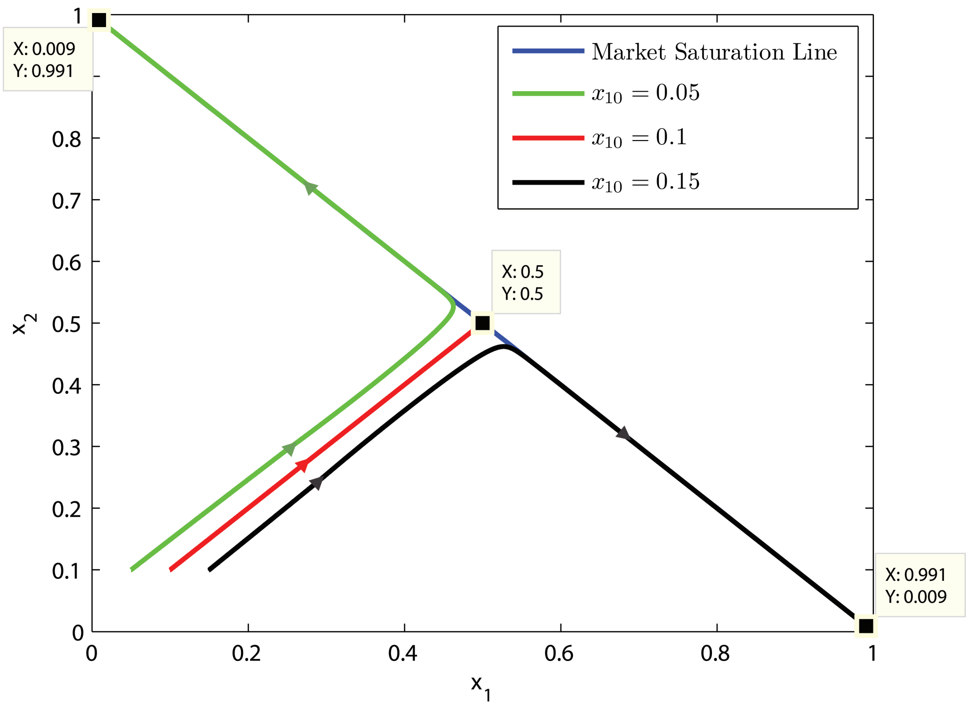

Figure 3.11: Trajectories for different values of x_{10}, while x_{20} is maintained constant at 0.1. The imbalance in the initial conditions determines the final outcome.

$$\frac{\partial u_{12}^{\dagger}}{\partial x_1} = \frac{(x_2 - m_2)\, c_{22} + (x_1 - 1 + m_2)\, \hat{c}_{11} - \hat{c}_{11}\,(1 - x_1 - x_2) - 2\,(x_1 - 1 + m_2) + u_{12}^{\dagger}}{x_2 - x_1 - 2m_2 + 1} \tag{3.34}$$

$$\frac{\partial u_{21}^{\dagger}}{\partial x_2} = \frac{(x_1 - m_1)\, c_{11} + (x_2 - 1 + m_1)\, \hat{c}_{22} - \hat{c}_{22}\,(1 - x_1 - x_2) - 2\,(x_2 - 1 + m_1) + u_{21}^{\dagger}}{x_1 - x_2 - 2m_1 + 1} \tag{3.35}$$

$$\frac{\partial u_{12}^{\dagger}}{\partial x_2} = \frac{(x_2 - m_2)\, c_{22} + (x_1 - 1 + m_2)\, \hat{c}_{11} - c_{22}\,(1 - x_1 - x_2) - 2\,(x_2 - m_2) - u_{12}^{\dagger}}{x_2 - x_1 - 2m_2 + 1} \tag{3.36}$$

$$\frac{\partial u_{21}^{\dagger}}{\partial x_1} = \frac{(x_1 - m_1)\, c_{11} + (x_2 - 1 + m_1)\, \hat{c}_{22} - c_{11}\,(1 - x_1 - x_2) - 2\,(x_1 - m_1) - u_{21}^{\dagger}}{x_1 - x_2 - 2m_1 + 1} \tag{3.37}$$

It is known from Lemma 2.2 that the equilibrium point must be located on the line $x_1 + x_2 = 1$. Hence we will make this substitution in the expressions above.

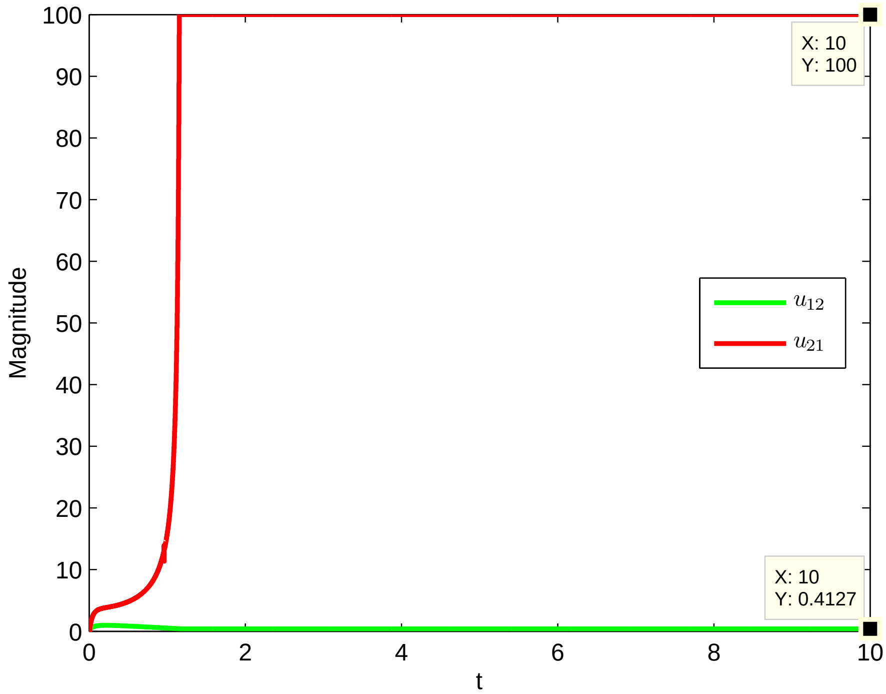

Figure 3.12: Control variables for $x_{10} = 0.15$. Notice that the predatory control of firm 1 reached its limit.

$$\left.\frac{\partial u_{12}^{\dagger}}{\partial x_1}\right|_{x=x_e} = \frac{c_{22} - \hat{c}_{11} + 1}{2} \tag{3.38}$$

$$\left.\frac{\partial u_{21}^{\dagger}}{\partial x_2}\right|_{x=x_e} = \frac{c_{11} - \hat{c}_{22} + 1}{2} \tag{3.39}$$

$$\left.\frac{\partial u_{12}^{\dagger}}{\partial x_2}\right|_{x=x_e} = \frac{c_{22} - \hat{c}_{11} - 1}{2} \tag{3.40}$$

$$\left.\frac{\partial u_{21}^{\dagger}}{\partial x_1}\right|_{x=x_e} = \frac{c_{11} - \hat{c}_{22} - 1}{2} \tag{3.41}$$

Substituting these expressions in (3.33) the following Jacobian will be obtained:

$$J = \begin{bmatrix} \frac{-c_{22}-c_{11}+\hat{c}_{11}-\hat{c}_{22}}{2} + k_1 + \hat{c}_{12} - 1 & \frac{-c_{22}-c_{11}+\hat{c}_{11}-\hat{c}_{22}}{2} - k_2 - \hat{c}_{21} + 1 \\ \frac{-c_{22}-c_{11}-\hat{c}_{11}+\hat{c}_{22}}{2} - k_1 - \hat{c}_{12} + 1 & \frac{-c_{22}-c_{11}-\hat{c}_{11}+\hat{c}_{22}}{2} + k_2 + \hat{c}_{21} - 11 \end{bmatrix} \tag{3.42}$$

In order to analyze the sign of the eigenvalues of matrix J, it is necessary evaluate both the determinant and the trace of this matrix.

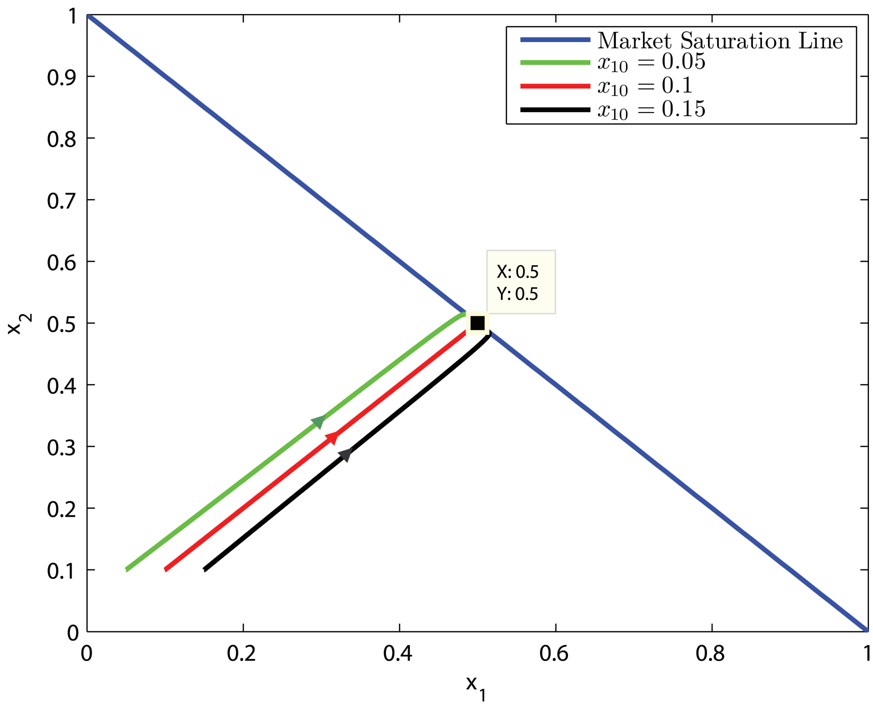

Figure 3.13: Trajectories using different values of x_{10} using $k_1 = k_2 = 0.5$, $m_1 = m_2 = 0.8$, $\hat{c}_{12} = \hat{c}_{21} = 0.4$, and $c_{11} = c_{22} = \hat{c}_{11} = \hat{c}_{22} = 3$. In this case, all trajectories converge to the point $(0.5, 0.5)$

$$det(J) = -(c_{22} + c_{11})(k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2) \tag{3.43}$$

$$tr(J) = -(c_{11} + c_{22}) + (k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2) \tag{3.44}$$

From equations (3.43) and (3.44) it is possible to notice that the eigenvalues of J are respectively $-(c_{11} + c_{22})$ and $k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2$. The first one is negative, since parameters c_{ii} were previously defined as positive numbers. Therefore if $k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2 > 0$ then the equilibrium point is a saddle, and if $k_1 + k_2 + \hat{c}_{12} + \hat{c}_{21} - 2 < 0$ the equilibrium point is a stable node. □

Theorem 3.6 allows us to affirm that, with identical parameters, the point $(0.5, 0.5)$ will be stable if and only if $k + c - 1 < 0$, where $k = k_1 = k_2$ and $c = \hat{c}_{21} = \hat{c}_{12}$. This result explains the differences between the simulation scenarios presented above. In the first one (Figure 3.11) $k + c - 1 = 0.5$, which resulted in unstable trajectories, and therefore a saturation in the control variable of the victorious firm. In the second scenario (Figure 3.13) $k + c - 1 = -0.1$, and the point $(0.5, 0.5)$ attracts all trajectories to itself, leading all initial conditions to a tie.

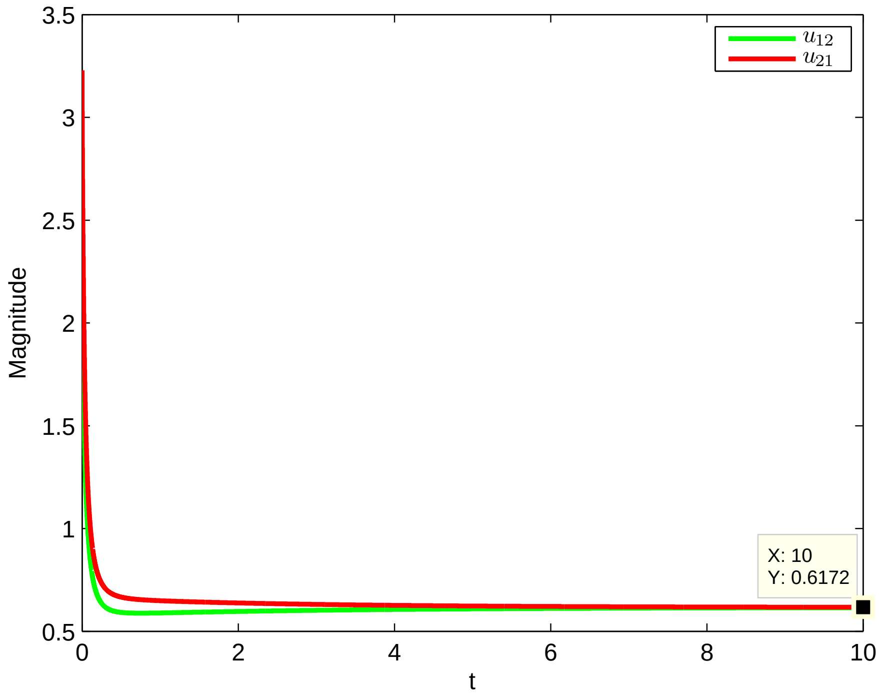

Figure 3.14: Control variables for $x_{10} = 0.15$. In this case, neither control variable reaches its saturation limit

It is also worth mentioning, that if $k_1 + k_2 + \hat{c}_{12} + \hat{c}_{12} - 2 = 0$ the system will have an infinite number of equilibrium points and their locus will be the line $x_1 + x_2 = 1$. In this very particular case, which is the case of null measure trajectory, the final point can only be calculated by the analytical expression for the trajectory. Since this is a very unlikely case we will not focus on it.

Going further in our analysis, if we look at equations (3.30) and the condition present in theorem 3.6 similar conclusions can be drawn if we desire to study the effect of each parameter imbalance. But we should keep in mind that the only possibilities are a stable equilibrium point or an unstable one.

From now, if we consider that every parameter in the model is fixed except $\hat{c}_{12}$ and $\hat{c}_{21}$, we can ask: what would be the best choice for them? We already know that depending on the region that the initial condition is located, one of the firms will conquer almost the entire market, if the condition present in theorem 3.6 sets the equilibrium point as a saddle. So, the best option for a firm is to augment its region, and it can do that by a choice of parameters that results in a favorable location for the unstable equilibrium point. For firm 1, the best place the equilibrium point can be located is at the point $(0, 1)$, since the trajectory that lies on the line $x_1 + x_2 = 1$ tends to move away from this point.

The best strategy for firm 1 is therefore to use a large value for $\hat{c}_{12}$, because when $\hat{c}_{12} \longrightarrow \infty$, $x_e \longrightarrow (0,1)$. An analogous strategy holds for firm 2, in regard to the point $(1,0)$ and the parameter $\hat{c}_{21}$.

Since, there is a limit to how large u_{ij} can be, the optimal choice for both firms is $\hat{c}_{ij} = u_{MAX}$. Since this particular choice is the best option for both firms using CLF control in a duopolistic market, we will further analyze it. For this choice of estimates, we will have a final market share $\mathbf{x}_f$ as follows:

$$\mathbf{x}_f = \left(\frac{u_{MAX} + k_2}{2u_{MAX} + k_1 + k_2}, \frac{u_{MAX} + k_1}{2u_{MAX} + k_1 + k_2} \right) \tag{3.45}$$

The expression (3.45) is obtained by simply replacing $\hat{c}_{ij}$ by u_{MAX} in equation (3.30). One should notice that if $u_{\max} \gg k_1$ and $u_{\max} \gg k_2$, the final market share $\mathbf{x}_f$ tends to $(0.5, 0.5)$. In figure 3.15 some simulations for different values of k_1 and k_2 are shown. The chosen values are not intended to represent real parameter values, but rather to emphasize the points being made.

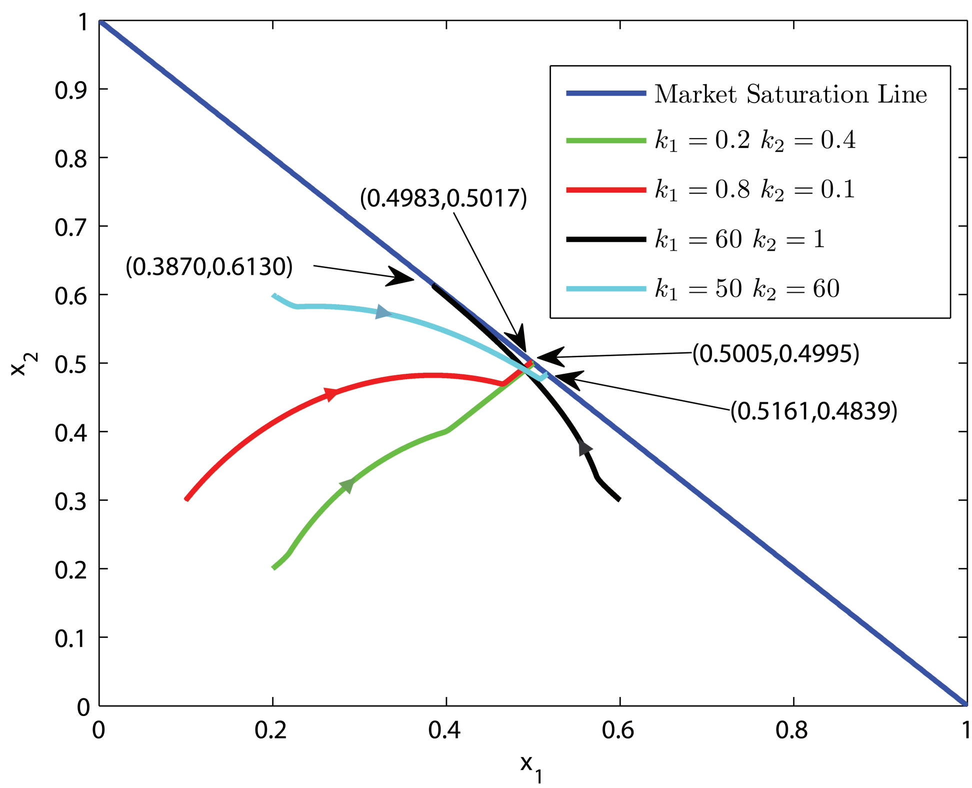

Figure 3.15: Examples of trajectories for different values of $\mathbf{x}_0$ and k_i. Notice that the larger the difference between k_1 and k_2, the farther the final state is located from the $(0.5, 0.5)$ point.

Unfortunately, keeping predation at a high level can be very costly for a firm. This is caused by the choices of estimates $\hat{c}_{ij}$. Since these estimates play a central role in this method, in the next chapter we will develop a better way to estimate them.

Chapter 4

Control design using CLF with estimator

In the previous chapter, we discovered that precise estimation of parameters c_{ij} is vital in order to track the desired final market share via CLF control. In this chapter, we will focus on the development of a new method to estimate the predation effort that a firm should make.

Initially, we will assume the point of view of firm 1 and we will try to estimate predation effort u_{12} made by firm 2. To begin our analysis, we need to look at equation (3.1) again, rewritten here for convenience.

$$\dot{x}_1 = u_{11}(1 - x_1 - x_2) - u_{12}x_1 + u_{21}x_2 - k_1x_1 + k_2x_2 \tag{4.1}$$

Since u_{21} has already been chosen (chapter 3), the functions $\dot{x}_1$ and u_{12} are the only unknown quantities in equation (4.1). Thus, if we can estimate $\dot{x}_1$, equation (4.1) can be rewritten in order to furnish an estimate of u_{12}. Indeed:

$$u_{12} = \frac{u_{11}(1 - x_1 - x_2) + u_{21}x_2 - k_1x_1 + k_2x_2 - \dot{x}_1}{x_1} \tag{4.2}$$

4.1 Robust differentiator using HOSM

The problem of obtaining a derivative $\dot{x}_1$ from a given function of time x_1 has already been studied in the past. The technique that will be used in this work can be found in [25], whose main results are stated briefly as follows.

In order to differentiate an unknown signal $f(t)$, the so called modified second order sliding algorithm can be considered as follows:

$$\begin{cases} \dot{x} = u \\ u = u_1 - \lambda |x - f(t)|^{\frac{1}{2}} \text{sign}(x - f(t)) \\ \dot{u}_1 = -\alpha \text{sign}(x - f(t)) \end{cases} \tag{4.3}$$

where α, $\lambda > 0$. Here $u(t)$ is the output of the differentiator, *i.e.*, $u(t)$ is the desired estimate of the derivative $\dot{x}$. In figure 4.1 it is possible to observe the block diagram that represents these equations.

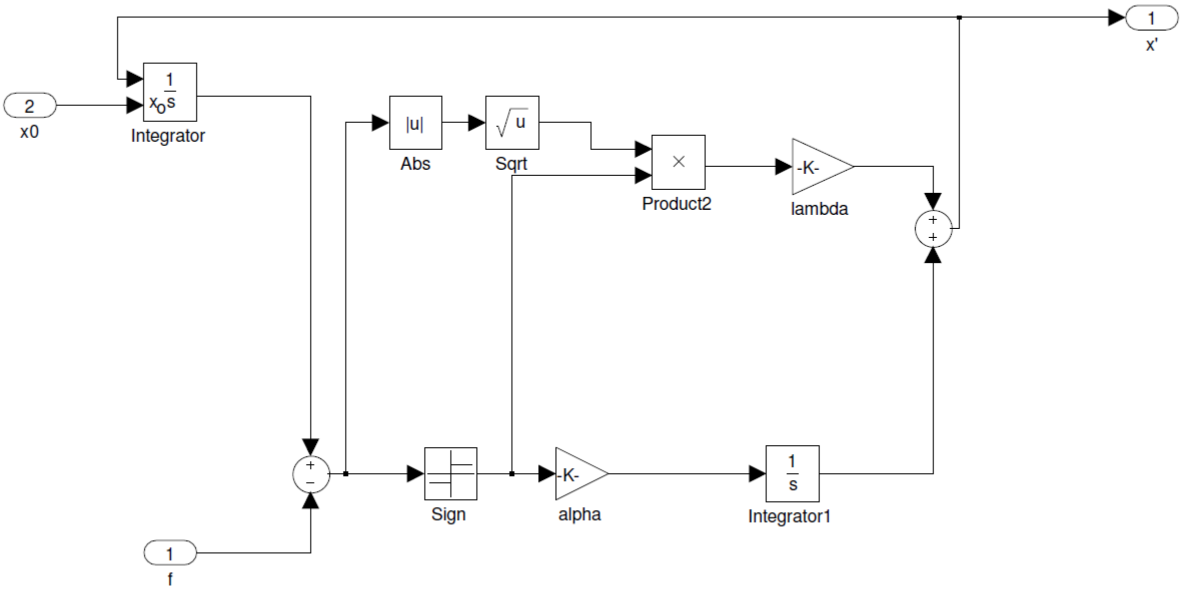

Figure 4.1: Block diagram for the equations of the modified second order sliding algorithm

Define the function $\phi(\alpha, \lambda, C) = \Psi(t_*)$, where (Σ, Ψ) is the solution of:

$$\begin{cases} \dot{\Sigma} = -|\Sigma|^{\frac{1}{2}} + \Psi \\ \dot{\Psi} = \begin{cases} -\frac{1}{\lambda^2}(\alpha - C) & , -|\Sigma|^{\frac{1}{2}} + \Psi > 0 \\ -\frac{1}{\lambda^2}(\alpha + C) & , -|\Sigma|^{\frac{1}{2}} + \Psi \leq 0 \end{cases} \end{cases} \tag{4.4}$$

with $\Sigma(0) = 0$, $\Psi(0) = 1$, $t_* = \inf\{t | t > 0 \text{ and } \Sigma(t) = 0 \text{ and } \Psi(t) < 0\}$. We state the main results for robust differentiation from [25]:

Theorem 4.1. *Let* $\alpha > C > 0$, $\lambda > 0$, $\phi(\alpha, \lambda, C) < 1$. *Then* $u(t)$ *becomes identically equal to* $\dot{f}(t)$ *after a finite time provided that* $\dot{f}$ *has a Lipschitz constant* C. *The smaller the value of* $\phi(\alpha, \lambda, C)$, *the faster the convergence.*

We also state the sufficient conditions:

$$\alpha > C, \qquad \lambda^2 \geq 4C\frac{\alpha + C}{\alpha - C} \tag{4.5}$$

Theorem 4.2. *If f, x, u_1 are measured at discrete times with time interval τ, and $f(t)$ has a derivative with Lipschitz constant $C > 0$, the inequality $\left|u(t) - \dot{f}(t)\right| < a\lambda^2\tau$ holds after a finite-time transient. Here $a > 0$ is some constant dependent on $\frac{\alpha - C}{\lambda^2}$ and $\frac{\alpha + C}{\lambda^2}$.*

The differentiator is also considered to be robust, since the maximum derivative error is also proportional to the square root of the input noise magnitude after a finite-time transient process. The demonstration of this statement and theorems 4.1 and 4.2 can be found in [25].

It is simple to notice from (4.1) that $-(u_{\max} + k_1) \leq \dot{x}_1 \leq u_{11} + u_{\max} + k_2$. Thus a Lipschitz constant for $\dot{x}_1$ can be estimated as:

$$C = \max\left(u_{\max} + k_1, u_{11} + u_{\max} + k_2\right) \tag{4.6}$$

This technique makes it possible to track $\dot{x}_1$, and consequently u_{12} by rewriting (4.1) as:

$$\hat{u}_{12} = \frac{\left(u_{11}(1 - x_1 - x_2) + u_{21}x_2 - k_1x_1 + k_2x_2 - \dot{\hat{x}}_1\right)}{x_1} \tag{4.7}$$

The attentive reader will notice that $\hat{u}_{12}$ is needed to compute u_{21} (Equation (3.11)) and vice-versa (Equation (4.7)). One way out of this circularity is to apply a small delay (sized as one simulation step) to $\hat{u}_{12}$ in the calculation of u_{21}.

Specifically, the use of (4.7) requires the knowledge of u_{21} at the exact instant that $\hat{u}_{12}$ is being estimated. In practice a delay of one simulation step, δ, in the value of $\hat{u}_{12}$ is used while calculating u_{21}:

$$u_{21}(t) = \frac{u_{21}^{\dagger} + (\hat{u}_{12}(t - \delta) + k_1)x_1}{x_2} - k_2 \tag{4.8}$$

with $u_{21}^{\dagger}$ defined in equation (3.12).

4.2 Single firm using CLF design and estimation

In this section we will present the results for a scenario in which firm 1 uses a CLF design together with an estimator based on the robust differentiator. For an initial analysis, firm 2 uses only a constant estimate for u_{21} and its predation effort is also constant. The other parameter values are the same as those used in figure 3.1.

As it is possible to see in figure 4.2, the error in the final market share point is drastically reduced. It also worth underscoring that the initial oscillating behavior in the state trajectory is due to the fact that the differentiator has always a transient. It must also be noticed that inequalities in (4.5) provide crude estimates. Smaller values for λ and α still achieve convergence, which is preferable since the convergence error depends on λ^2 (4.2). Also, we will not be concerned with the values of λ and α. Given that they are adequate for convergence, they only affect the transient, and our main focus is on permanent regime analysis.

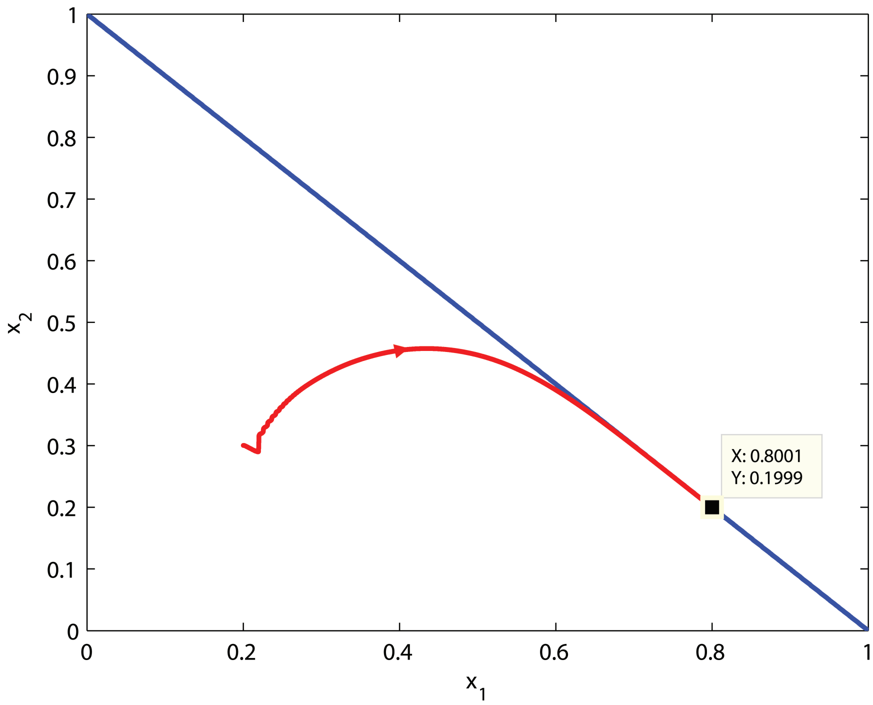

Figure 4.2: State trajectory for $\alpha = \lambda = 10$. These values were found experimentally and do not obey inequations in (4.5), which results in smaller errors. Moreover, the initial oscillation is due to differentiation transient

Furthermore, figure 4.3 shows us that the estimator was able to track u_{12} with great precision. The minor errors that are present in both final state $\mathbf{x}_f$ and estimate of firm 2's predation effort $\hat{u}_{12}$ are due to the error present in the robust differentiator. Some undesirable spikes also appear in both estimate and control signal. This problem will be discussed later in Section 4.4.

The next result, concerns a scenario in which firm 2 uses a CLF design but does not use an estimator. All other market parameters were kept the same as the previous case. Figures 4.4 and 4.5 show an example of the intuitive fact that having a dynamical estimator is a better tactic than using a static estimate. Since u_{21} accesses a precise estimate, it develops a better response to firm 2's behavior. As discussed previously in

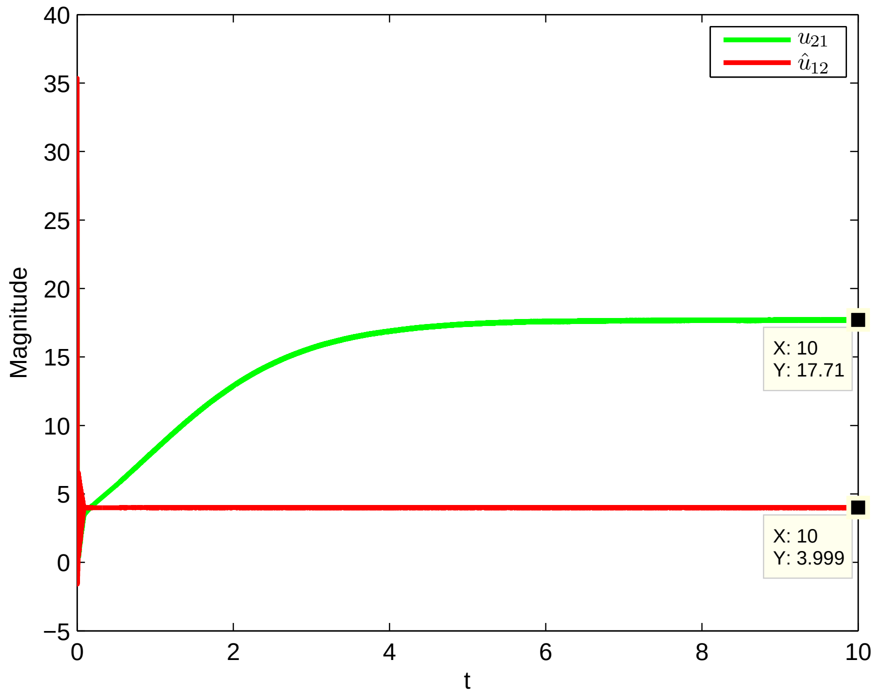

Figure 4.3: Control signal u_{21} and estimate $\hat{u}_{12}$. In this case, u_{12} is equal to $c_{12} = 4$. Some spikes appear in the transient response due to the characteristics of Levant's differentiator.

section 3.3, if a firm underestimates its opponent's predation effort, it will end up having a smaller market share than its desired market share. Then, firm 1 just needs to adjust its predation effort in order to create a sufficient gap between u_{21} and $\hat{u}_{21}$, so that firm 2 will be underestimating its competitor. Ultimately the expression for the equilibrium utilizes firm 1's desired market share point. The expression for steady state value of u_{21} is as follows:

$$u_{21f} = \frac{m_2 - x_{2f}}{x_{2f}} + \hat{c}_{21} \tag{4.9}$$

The proof for equation (4.9) is similar to that for equation (3.15). Finally, it is worth mentioning that the only disadvantage of this technique is a minor error in the final state. This steady state error is due to a minor error present in the estimator. It is possible to find this error if we check the steady state equation for x_1:

$$0 = -u_{12}x_{1f} + u_{21}x_{2f} - k_1x_{1f} + k_2x_{2f} \tag{4.10}$$

substituting u_{21} as in equations (3.11) and (3.12) leads to:

$$u_{12}x_{1f} = -(x_1 - m_1) + \hat{u}_{12}x_{1f} \tag{4.11}$$

which becomes:

$$x_{1f} = \frac{m_1}{1 - \epsilon} \tag{4.12}$$

where $\epsilon = u_{12} - \hat{u}_{12}$

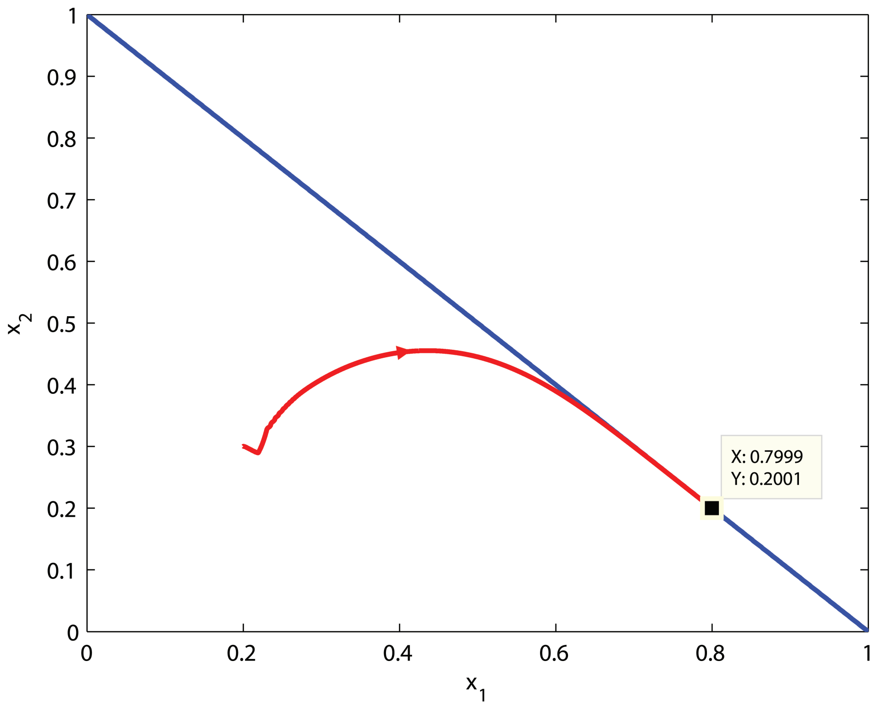

Figure 4.4: State trajectory for $\alpha = \lambda = 10$. These values do not satisfy the sufficient conditions, but still result in convergence. The initial segment of the trajectory is a consequence of the transient of the estimator. The final state has a small error.

4.3 Both firms using CLF design and estimator

In this section the problem of both firms using a control via CLF, along with a real time estimator based on the robust differentiator, will be presented. To begin this analysis, we will check whether the technique achieves the 50-50 market share point under identical initial conditions. In figure 4.6, it is possible to see that no firm prevails if both use the same tactics. The predation effort each firm uses, quickly grows to its maximum value, as can be seen in figure 4.7, in which control signal u_{12} and both estimates are omitted, since they do not differ visibly from control signal u_{21}.

We will approach the problem of having an unbalanced market scenario. Initially we will change only the initial condition and check what happens to the trajectory. Figure

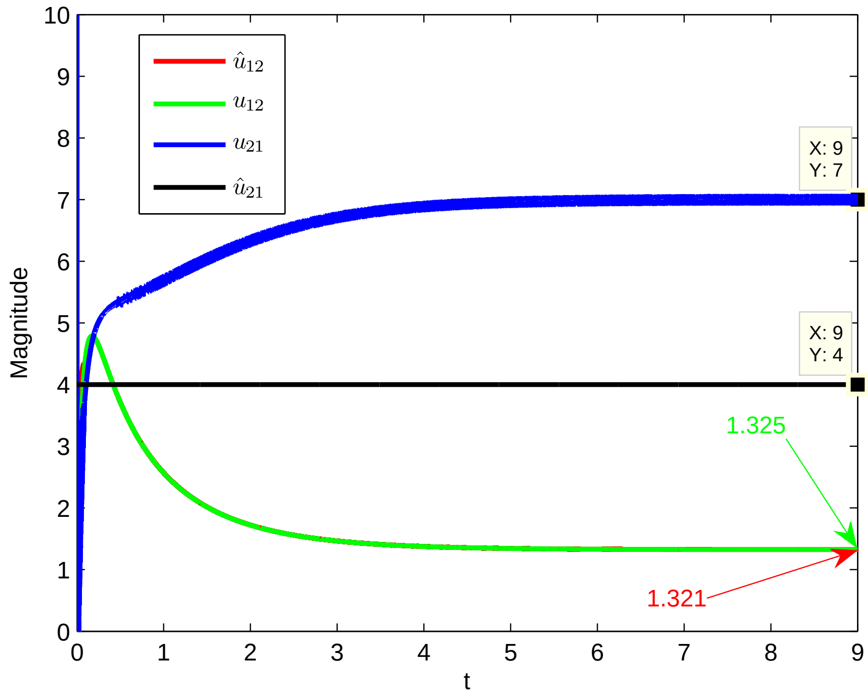

Figure 4.5: Estimates $\hat{u}_{12}$ and $\hat{u}_{21}$ and control signals u_{12} and u_{21}. The estimator final value has a small error. The spike is also present in the control signal.

4.8 reveals a surprising outcome: firm 1 and 2 ended up sharing the market equally. An explanation for this fact can be guessed from figure 4.9, where both predation efforts, u_{12} and u_{21}, appear. Initially, both firms use a rapidly increasing predatory effort in order to achieve their goals. Since firm 1 started with a bigger market share, it has to expend more effort to maintain it, due to the effect of churn. This explains why firm 1's predation effort is always bigger than firm 2's predation effort, that is, $u_{21} \geq u_{12}$. At the time u_{21} reaches the limit $u_{\max}$, u_{12} starts to develop a trajectory that will lead firm 2 to its objective at $m_2 = 0.8$, since firm 1 is stuck with a constant predation. But in order to achieve this objective, a predation effort with magnitude much greater than the limit imposed by $u_{\max}$ is required. So, when u_{12} reaches the value $u_{\max}$, both predation efforts become equivalent. This occurs exactly at the moment that the state trajectory crosses the line $x_1 = x_2$.

Now that we have studied the effects of different initial conditions, we turn to other market parameters. In figure 4.10, several state trajectories are presented with different parameter values, as shown in table 4.1. Churn parameters are kept constant for now, since we will carefully discuss them later on.

As can be seen, variations in the model parameters are only responsible for changes in the concavity of the state trajectory. The property of achieving the equal market share

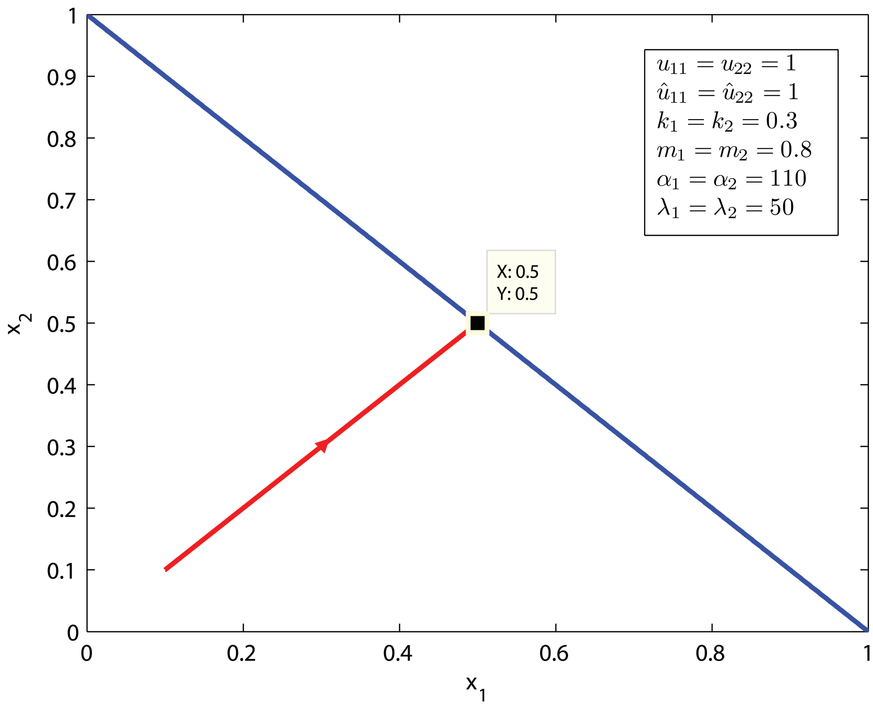

Figure 4.6: Trajectory for an even market and equal initial conditions. A tie is always expected when all parameters and control strategies are the same.

point was not lost, and the technique has revealed to be itself robust with respect to these parameters, since the steady state dynamics do not depend on them. Also it is noticeable that the trajectory follows the line $x_1 = x_2$ after a certain time, and then converges to the equilibrium point $(0.5, 0.5)$.

Trajectory Number	u_{11}	u_{22}	$\hat{u}_{11}$	$\hat{u}_{11}$	x_{10}	x_{20}	m_1	m_2
1	2	0.1	2	1	0.3	0.1	0.8	0.9
2	2	1	1	1.5	0.1	0.25	0.9	0.7
3	1	3	2	2	0.2	0.3	0.85	0.8
4	4	2	3	3	0.4	0.15	0.7	0.95
5	0.5	0.5	1	0.3	0.05	0.1	1	1
6	1	10	4	2	0.2	0.05	1	0.8

Table 4.1: Table of parameters used in simulations

We have postponed the discussion for churn parameters because, the final equilibrium point depends on them. The final market share point expression is the same as the one shown in equation (3.45). We rewrite it here for convenience.

$$\mathbf{x}_f = \left(\frac{u_{MAX} + k_2}{2u_{MAX} + k_1 + k_2}, \frac{u_{MAX} + k_1}{2u_{MAX} + k_1 + k_2} \right) \tag{4.13}$$

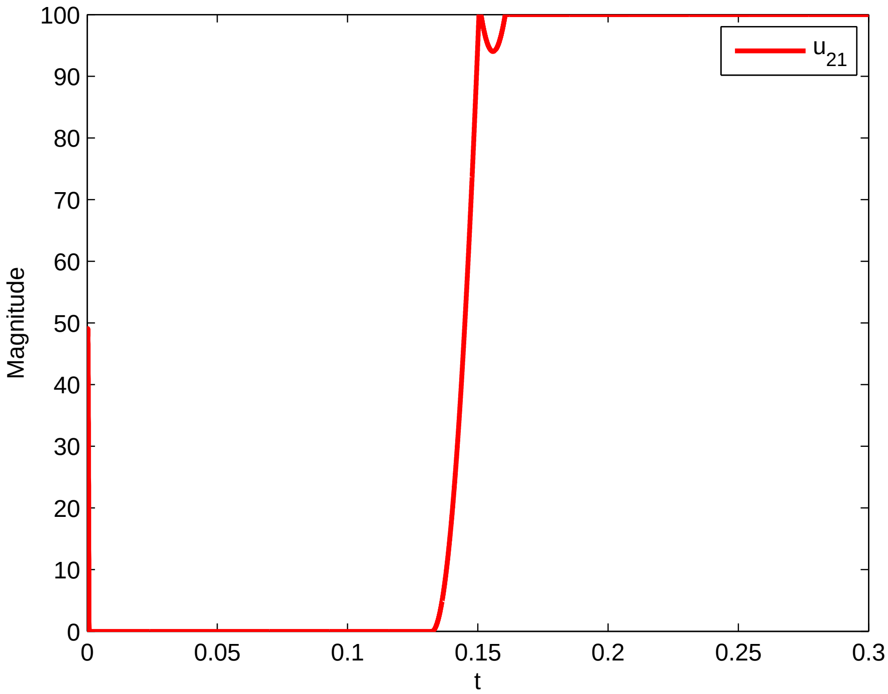

Figure 4.7: Predation effort u_{21}. Again a small spike is present. Notice that the control quickly saturate.

Figure 4.11 shows some state trajectories for different values of k_1 and k_2, and figure 4.12 shows their respective time plots. As it is possible to see experimentally from figure 4.11 and analytically from equation (4.13), the use of the technique by both firms simultaneously could not avoid the churn effects, let alone achieve their desired market shares. Also, the bigger the maximum value of predation effort, the more independent from churn parameters the equilibrium state becomes. This is easy to check since if $u_{\max} \gg k_1$ and $u_{\max} \gg k_2$, then $u_{\max} + k_1 \approx u_{\max}$ and $u_{\max} + k_2 \approx u_{\max}$, and as a result the final state $\mathbf{x}_f$ tends to $(0.5, 0.5)$. In the next section we will further discuss the inherent limitations of the proposed technique.

4.4 Limitations of the technique

In this subsection we discuss the limitations of the proposed technique, as well as their consequences. The first observation is that discontinuities may be present in the control variable. Let us recall expression (3.11):

$$u_{21} = \frac{u_{21}^{\dagger} + (u_{12} + k_1)x_1}{x_2} - k_2 \tag{4.14}$$

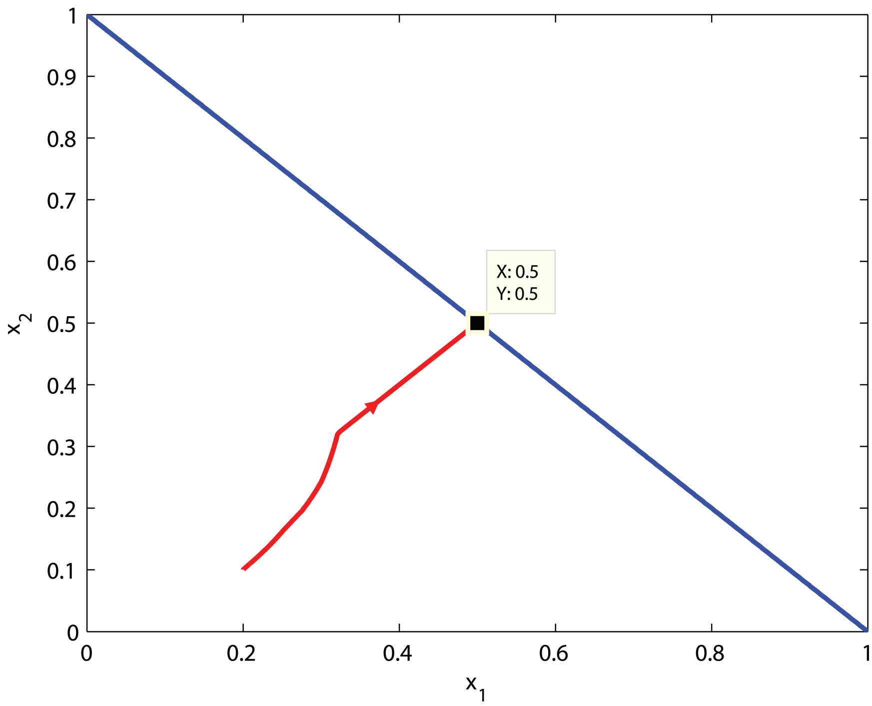

Figure 4.8: Trajectory for an even market and different initial conditions. Despite the inbalance on the initial conditions, firms 1 and 2 ended up evenly sharing the market

where

$$u_{21}^{\dagger} = \frac{-[(x_1 - m_1)c_{11} + (x_2 - 1 + m_1)c_{22}](1 - x_1 - x_2) - (x_1 - m_1)^2 - (x_2 - (1 - m_1))^2}{a} \tag{4.15}$$

$$a = x_1 - x_2 - 2m_1 + 1 \tag{4.16}$$

As it is possible to see a defines a line in the plane, and also is the denominator in the expression of $u_{21}^{\dagger}$. An undesirable discontinuity will occur if a trajectory reaches any point on this line. The consequences are depicted in figures 4.13 and 4.14, which show, respectively, the trajectory and control effort for a situation in which this line is crossed. As one can see in figures 4.13 and 4.14, the trajectory does not appear to be affected. On the other hand, both control signals present a severe discontinuity.

When the trajectory approaches the line, the value of u_{21} approaches ∞. This phenomenon is minimized in our simulation since we have imposed an upper bound on the predation efforts. The real problem shows up when the trajectory reaches the other side of the discontinuity line, and the value of u_{21} suddenly turns to $-\infty$, but is set to zero. Firm 2's rapidly detects this behavior and zeros its predation effort, since firm 1 is no longer

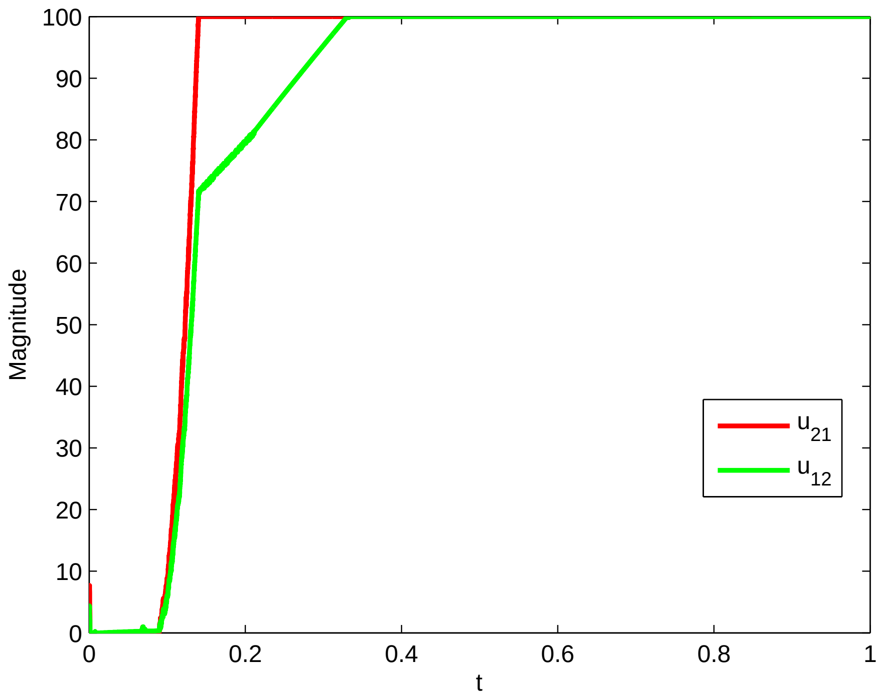

Figure 4.9: Control Signals u_{21} and u_{12}. Notice that after control u_{21} saturates, firm 2 slows down its predation effort since firm 1 is making constant effort. And later, control u_{12} also saturates.

making any predation effort. In the end, both firms end up having a major discontinuity that is not feasible in a real market scenario.

If both firms are aggressively competitive (*i.e.*, both m_1 and m_2 are equal to 1) then the line $a = 0$ does not divide the domain of possible market shares ($0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$), and therefore, no trajectory will be able to cross the line $a = 0$ and no discontinuity in the control variable will occur.

Another difficulty one may encounter while applying this technique is the presence of spikes in the control signal, that has already been cited before. Figure 4.15 shows an example of spike in a control signal. As mentioned before, large spikes in control effort (=expenditure) are generally infeasible for a firm in a real market scenario, which makes this behavior undesirable.

Since this problem is mainly generated by the transient of the robust differentiator, we will use a method similar to the one proposed by Levant [25] in order to reduce the transient. A low pass first order filter described by equation (4.17), with an adequately chosen τ, will be introduced at the output of the robust differentiator.

$$H(s) = \frac{1}{\tau s + 1} \tag{4.17}$$

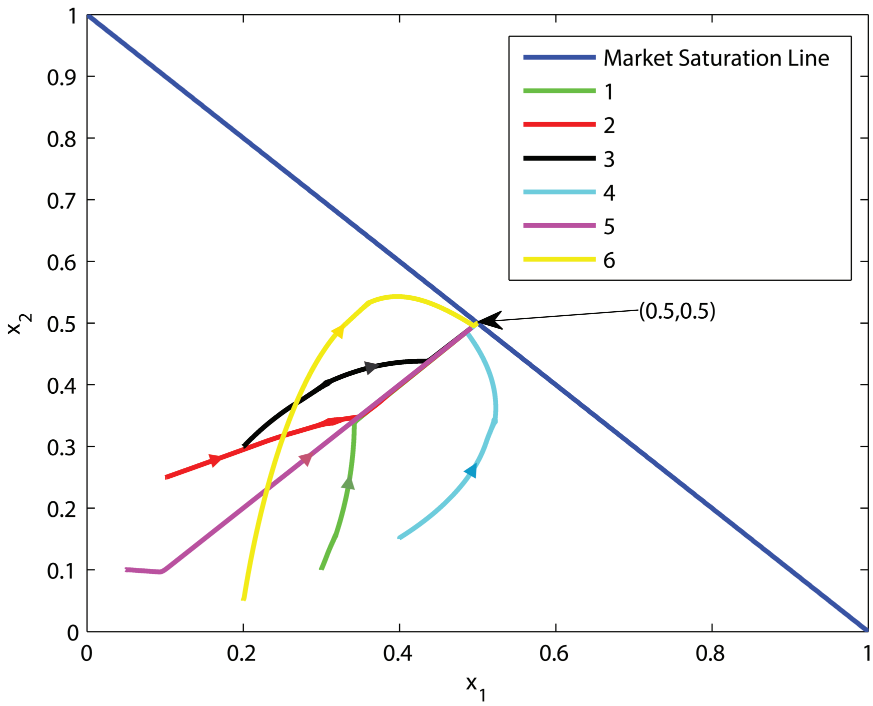

Figure 4.10: Examples of state trajectories with $k_1 = k_2 = 0.3$. Notice that point $(0.5, 0.5)$ is the unique equilibrium point for equal churn parameters. Other parameters only affect the transient response.

As we can see in figure 4.16, the use of the filter causes disappearance of spiking behavior. The control signal now starts at a higher magnitude and keeps increasing from there. It is also worth mentioning, that if this filter is used in a situation in which only one firm is using the proposed control scheme, a small error will appear in the steady state, proportional to τ.

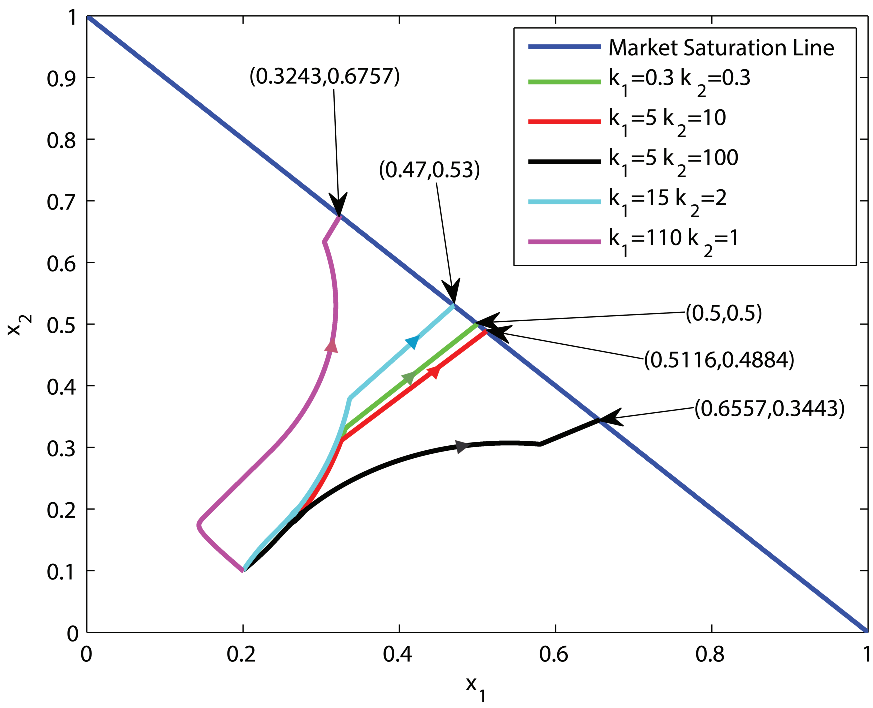

Figure 4.11: Example of trajectories for different values of k_1 and k_2. The final state depends only on churn parameters and predation effort saturation level.

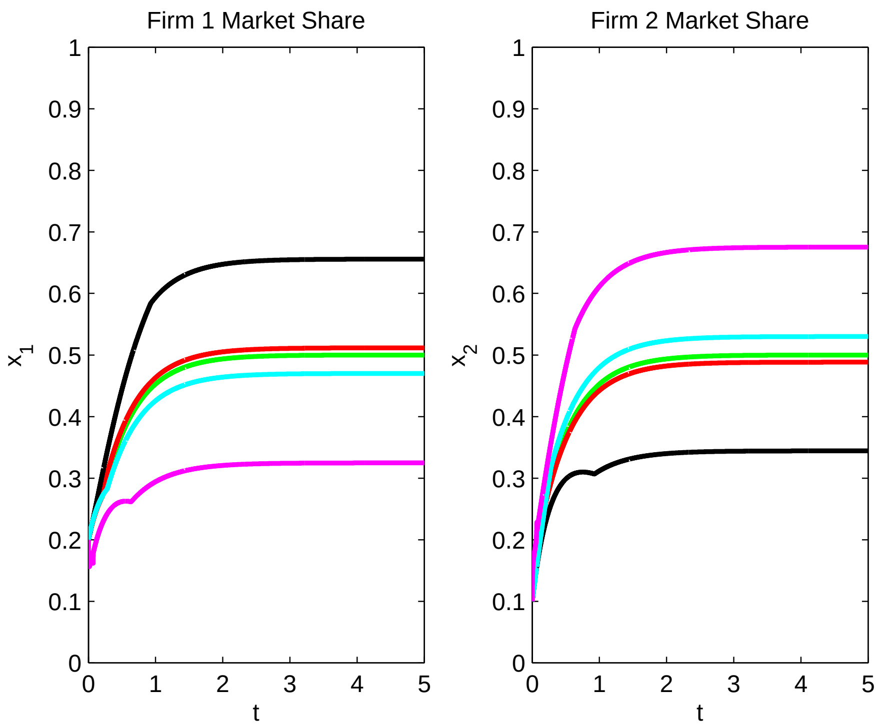

Figure 4.12: Firm 1 and 2 market shares along the time for different values of the churn parameters.

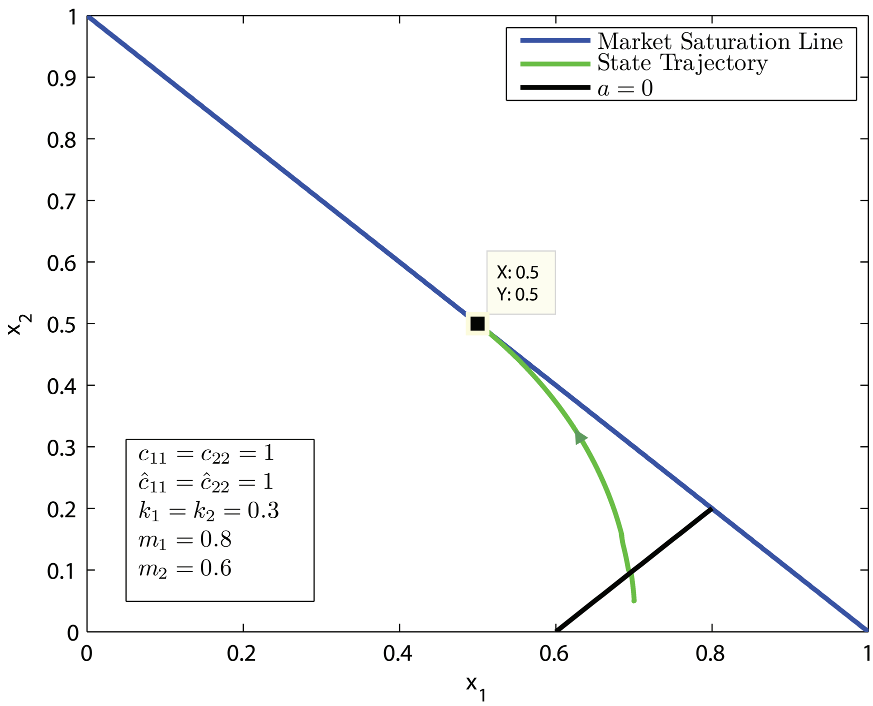

Figure 4.13: State trajectory crossing the discontinuity line. The trajectory is unaltered after crossing it.

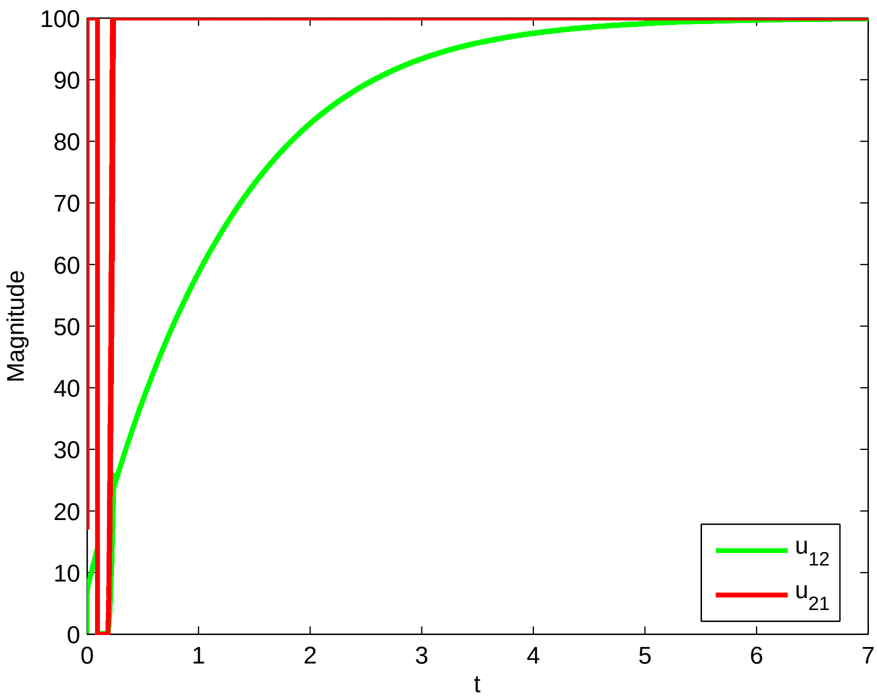

Figure 4.14: Control signals now have discontinuities. Notice that the discontinuity occurs exactly when the trajectory crosses the line $a = 0$. This is an undesired effect.

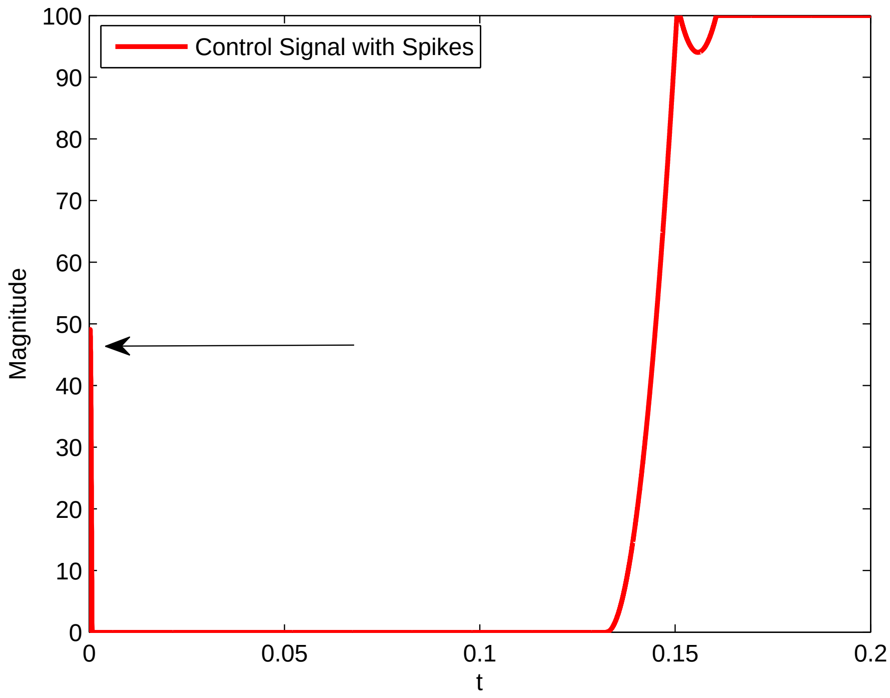

Figure 4.15: Control signal with spikes. A filter is necessary to reduce the latter.

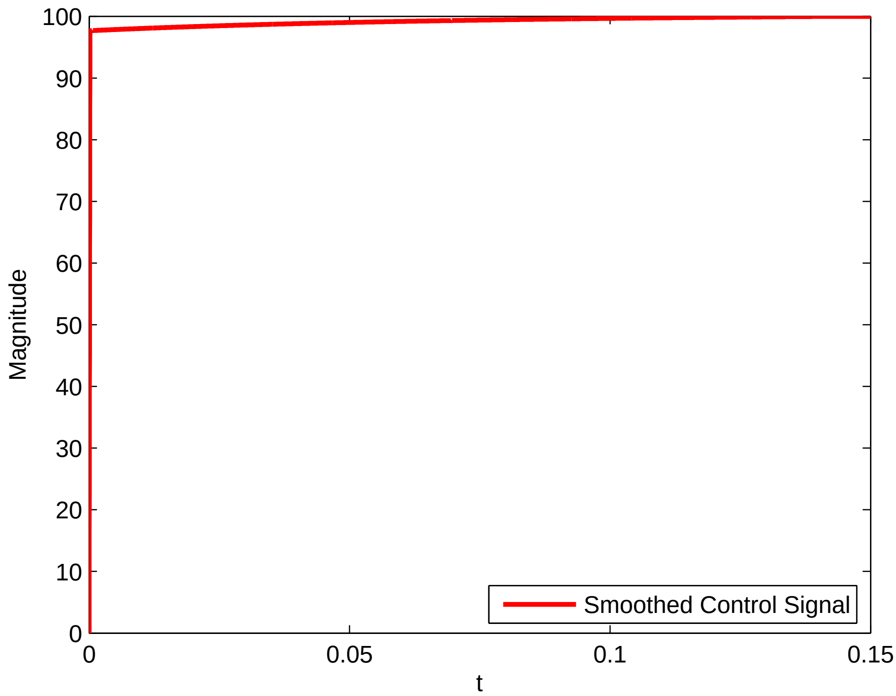

Figure 4.16: Control signal smoothed with $\tau = 0.1$. The spike is gone, and the control rises more smoothly.

Chapter 5

Conclusions and future works

In this section we present a brief review of the results obtained through this work, as well as some conclusions about them. Subsequently, some ideas for future work are proposed.

5.1 Conclusions

In this dissertation we approached the problem of controlling the duopoly model with churn proposed in [24], which is based on the classical Vidale-Wolfe model. A CLF control for predatory advertising was designed (a brief review of CLF method was presented in section 3.1) and used, along with a constant positive advertising, and many different scenarios were studied.

In section 3.2 the problem of a single firm using CLF predatory control was studied, and, with the aid of theorem 3.3, we discovered that as long as a firm has an accurate estimate of the predatory control of its competitor, any final objective can be achieved. Also it is required that no restrictions are imposed on the magnitude of the firm's control signal.

Section 3.3 answered the question of the role played by the estimate of the competitor's predatory control. Equation (3.22) shows that an inaccurate estimate can compromise the effectiveness of the CLF considerably.

Section 3.4 studied the problem of both firms using a CLF strategy. After some simulations, we verified, and latter demonstrated (theorem 3.6), that two scenarios are possible for the equilibrium point (if it is unique): a stable node equilibrium, to which all trajectories in the plane are attracted, and a saddle. The latter causes the trajectories to deviate from it leading them either to the point $(0, 1)$ or $(1, 0)$, depending on the initial condition.

The results fo chapter 3 imply that a firm should estimate the predatory effort of its competitor as accurately as possible, in order to meet the condition presented in theorem

3.6 to obtain a saddle point and conquer almost the entire market.

Since the estimation of the competitor's predatory control plays such a key role, chapter 4 was dedicated to find a better alternative to the estimation problem. Section 4.1 made a brief presentation of Levant's differentiator [25], which was the theoretical basis for the estimator proposed in this dissertation.

Section 4.2 presented the simulations of one firm using the estimator based on the robust differentiator, and its competitor using a CLF with constant estimate. We verified that the final objective was attained with a small error, proportional to the error of the robust differentiator.

A scenario in which both firms use the estimator was simulated next. We concluded that the equilibrium point achieved by the system is dependent on the churn parameters and the maximum value imposed on the predatory control, as shown in equation (4.13).

Finally, section 4.4 investigated the limitations of the technique utilized. We explained that a severe discontinuity can occur in the control signal (Figure 4.14), when a trajectory crosses the line that zeroes the denominator of the expression for the predatory control (Figure 4.13). Also, when utilizing a robust differentiator, spikes can occur in the transient of the control signal (Figure 4.15, which is a very undesirable phenomena. In order to diminish this effect, a simple low pass filter was proposed in (4.17).

After this brief review of the results obtained in this dissertation, it is possible to conclude that the use of a CLF predatory control proved to be much more efficient than a constant one. However, the technique is very susceptible to errors in the estimate of the competitor's predatory control. Also, the control signal can present severe discontinuities depending on the objective market share.

If both firms are using CLF strategy, the best choice for a firm to estimate its competitor's predation effort is to choose the maximum control effort possible. Furthermore, if an estimator based on a robust differentiator is available, it should be used, since its combined use with the CLF technique was more efficient than the use of a constant estimate. One should be aware that the robust differentiator can introduce spikes in the marketing expenditure of a firm, which can be very undesirable in a real context.

Through this work, it was possible to observe, the CLF technique stressed the model to its limits. Many saturation in the control signals were found, and the more we developed the control technique, the more the control signal resembled a step signal. Expending the entire budget of a firm for an indefinitely long time in a real market scenario surely is not the best answer one can obtain. This observation exposes a fragility in the model, concerning the predation term in the dynamics. It is not feasible or trustworthy to rely on a predation effort that achieves market share directly proportional to the predation expenditure.

5.2 Future works

For a future work, a change in predation dynamics should definitely be modeled. It should be developed in a way that saturation does not occur so easily. The predation effort should be related to the variation of market share by a bounded function (not linear, as it is as of now).

Improvements to the CLF technique using more sophisticated functions, should also be researched in order to mitigate, or even eliminate, discontinuities, spikes, and other undesirable effects.

Also, studies considering game theory, and thus considering the system as a differential game, may rise more information about optimality as well as minimax strategies for a firm.

Bibliography

[1] DORFMAN, R., STEINER, P. O. "Optimal Advertising and Optimal Quality", *American Economic Review*, v. 44, pp. 826–836, 1954.

[2] VIDALE, M. L., WOLFE, H. B. "An operations research study of sales response to advertising", *Operations Research*, v. 5, pp. 370–381, 1957.

[3] OZGA, S. A. "Imperfect Markets through Lack of Knowledge", *The Quarterly Journal of Economics*, v. 74, n. 1, pp. 29–52, 1960.

[4] LANCHESTER, F. W. *Aircraft in warfare.* New York, Appleton, 1916.

[5] LANCHESTER, F. W. "The world of mathematics". v. 4, cap. Mathematics in warfare, pp. 2138–2157, New York, Simon and Schuster, 1956. ed. J. R. Newman, republished by Dover in 2000.

[6] KIMBALL, G. E. "Some industrial applications of military operations research methods", *Operations Research*, v. 5, n. 2, pp. 201–204, 1957.

[7] LITTLE, J. D. C. "Aggregate advertising models: The state of the art", *Operations Research*, pp. 629–667, 1979.

[8] CHINTAGUNTA, P., VILCASSIM, N. J. "An empirical investigation of advertising strategies in a dynamic duopoly", *Management Science*, v. 38, n. 9, pp. 1230–1244, 1992.

[9] DAS, B. J., CHAPPELL, W. F., SHUGHART II, W. F. "Advertising, competition and market share instability", *Applied Economics*, v. 25, pp. 1409–1412, 1993.

[10] WANG, Q., WU, Z. "An empirical study on the Lanchester model of combat for competitive advertising decisions", *European Journal of Operational Research*, v. 183, pp. 871–881, 2007.

[11] NERLOVE, M., ARROW, K. J. "Optimal advertising policy under dynamic conditions", *Economica*, v. 29, pp. 129–142, 1962.

[12] SETHI, S. P., THOMPSON, G. L. *Optimal Control Theory: Applications to Management Science and Economics*. Springer-Verlag, 2000.

[13] WANG, Q., WU, Z. "A duopolistic model of dynamic competitive advertising", *European Journal of Operational Research*, v. 128, pp. 213–226, 2001.

[14] ERICKSON, G. M. *Dynamic Models of Advertising Competition*. second ed. Dordrecht, Kluwer Academic Publishers, 2003.

[15] QI, J., WANG, D.-W. "Optimal Control Strategies for an Advertising Competing Model", *System Engineering - Theory and Practice*, v. 27, n. 1, pp. 39–44, 2007.

[16] HUANG, J., LENG, M.-M., LIANG, L.-P. "Recent developments in dynamic advertising research", *European Journal of Operational Research*, v. 220, pp. 591–609, 2012. Disponível em: <http://dx.doi.org/10.1016/j.ejor.2012.02.031>.

[17] JORGENSEN, S. "A Survey Of Some Differential Games In Advertising", *Journal of Economic Dynamics and Control*, v. 4, pp. 341–369, 1982.

[18] FEICHTINGER, G., JORGENSEN, S. "Differential game models in management science", *European Journal of Operational Research*, v. 14, pp. 137–155, 1983.

[19] BISCHI, G. I., NAIMZADA, A. "Advances in Dynamic Games and Applications". v. 5, cap. Global Analysis of a Dynamic Duopoly Game with Bounded Rationality, pp. 361–385, Birkhauser, 1999.

[20] BISCHI, G. I., CHIARELLA, C., KOPEL, M., etal. *Nonlinear Oligopolies: Stability and Bifurcations*. Berlin, Springer-Verlag, 2010.

[21] JONES, P. C. "Analysis Of A Dynamic Duopoly Model Of Advertising", *Mathematics of Operations Research*, v. 8, n. 1, pp. 122–134, Feb 1983.

[22] SICE, P., MOSEKILDE, E., MOSCARDINI, A., etal. "Using system dynamics to analyse interactions in duopoly competition", *System Dynamics Review*, v. 16, n. 2, pp. 113–133, 2000.

[23] PRASAD, A., SETHI, S. P. "Competitive Advertising Under Uncertainty: A Stochastic Differential Game Approach", *Journal of Optimization Theory and Applications*, v. 123, pp. 163–185, 2004. ISSN: 0022-3239. doi: 10.1023/B:JOTA.0000043996.62867.20.

[24] KASZKUREWICZ, E., BHAYA, A. "A study of duopolistic dynamics with competitive advertising based on state-dependent switching behavior". In: *Proceedings of the* 51$^{\text{st}}$ *IEEE Conference on Decision and Control (CDC), Maui, December 2012*, pp. 7081–7087, 2012.

[25] LEVANT, A. "Robust exact differentiation via sliding mode technique", *Automatica*, v. 34, n. 3, pp. 379–384, March 1998.

[26] MUTANEN, T. *Customer churn analysis - a case study.* Relatório Técnico VTTR0118406, VTT, 2006.

[27] WEN, D.-H., CHEN, R.-B., LU, T.-J. "Factors for wireless operators' collusion and competition", *The Journal of China Universities of Posts and Telecommunications*, v. 14, n. 1, pp. 61–64, 2007.

[28] HE, X.-L., KRISHNAMOORTHY, A., PRASAD, A., etal. "Retail competition and cooperative advertising", *Operations Research Letters*, v. 39, n. 1, pp. 11 – 16, 2011. ISSN: 0167-6377. doi: 10.1016/j.orl.2010.10.006. Disponível em: <http://www.sciencedirect.com/science/article/pii/S0167637710001446>.

[29] PRASAD, A., SETHI, S. P., NAIK, P. A. "Understanding the impact of churn in dynamic oligopoly markets", *Automatica*, v. 48, n. 11, pp. 2882–2887, nov. 2012. ISSN: 0005-1098. doi: 10.1016/j.automatica.2012.08.031. Disponível em: <http://dx.doi.org/10.1016/j.automatica.2012.08.031>.

[30] KHALIL, H. K. *Nonlinear Systems.* New Jersey, Prentice Hall, 2002.