OPTIMAL CONTROL OF THE VIDALE-WOLFE-DEAL AND THREE POPULATIONS MODELS OF MARKET SHARE DYNAMICS

Luiz Carlos de Barros Roth

Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Engenharia Elétrica, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Engenharia Elétrica.

Orientador: Amit Bhaya

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DISSERTAÇÃO SUBMETIDA AO CORPO DOCENTE DO INSTITUTO ALBERTO LUIZ COIMBRA DE PÓS-GRADUAÇÃO E PESQUISA DE ENGENHARIA (COPPE) DA UNIVERSIDADE FEDERAL DO RIO DE JANEIRO COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE MESTRE EM CIÊNCIAS EM ENGENHARIA ELÉTRICA.

Examinada por:

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Prof. Amit Bhaya, Ph.D.

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Prof. Eugenius Kaszkurewicz, D.Sc.

_____

Prof. Nelson Maculan Filho, D.Sc.

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To God and my parents.

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CONTROLE ÓTIMO DOS MODELOS DE DINÂMICAS DE MERCADO DE VIDALE-WOLFE-DEAL E TRÊS POPULAÇÕES

Luiz Carlos de Barros Roth

Março/2018

Orientador: Amit Bhaya

Programa: Engenharia Elétrica

O propósito desta dissertação é estudar a resposta ótima dos modelos de venda-publicidade em duopólios utilizando modernos software de otimização (JModel-ica.org e JuMP). Especificamente, os modelos adotados foram Vidale-Wolfe-Deal e Três Populações (um modelo do tipo Lotka-Volterra). As análises de duopólio são divididas em duas partes: uma, que soluciona o problema de controle ótimo com o objetivo de maximizar o lucro líquido de ambas as empresas de uma só vez, referido como cooperação simultânea, e outra, que coloca as duas empresas como oponentes em um jogo sequencial, cada uma com o objetivo individual de maximizar o lucro líquido durante seu turno, referido como competição sequencial.

As contribuições da dissertação são: prover análise de estabilidade do modelo Vidale-Wolfe-Deal mostrando que qualquer controle que atinja valores constantes positivos leva a um equilíbrio estável das fatias de mercado, propor uma versão do modelo de Três Populações para o duopólio, e, por fim, propor e solucionar um jogo sequencial baseado em iterações de Líder-Seguidor para o modelo Vidale-Wolfe-Deal.

Abstract of Dissertation presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Master of Science (M.Sc.)

OPTIMAL CONTROL OF THE VIDALE-WOLFE-DEAL AND THREE
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Luiz Carlos de Barros Roth

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Advisor: Amit Bhaya

Department: Electrical Engineering

The purpose of this dissertation is to study the optimal response of sales-advertising models for duopolies using modern optimization software (JModelica.org and JuMP). Specifically, the models adopted were the Vidale-Wolfe-Deal and the Three Populations, a Lotka-Volterra type model. Duopoly analysis is split in two parts: one that solves an optimal control problem with the objective of maximizing the net profit of both firms in one-shot, referred to as simultaneous co-operation, and the other that places the two firms as opponents in a sequential game, each with the individual goal of maximizing net profit on its turn, referred to as sequential competition.

The contributions of this dissertation are: providing a stability analysis for the Vidale-Wolfe-Deal model, which shows that any control attaining final constant positive values leads to a stable equilibrium of market shares, proposing a duopolistic version of the Three Populations model, and, lastly, proposing and solving a sequential game based on Leader-Follower iteration for the Vidale-Wolfe-Deal model.

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Chapter 1

Introduction

A market is any medium driven by the laws of supply and demand through which buyers and sellers can exchange goods and services. A market does not need to be a physical location. Auction sites and on-line shopping are examples of electronic transactions that can take place without any party ever meeting in person. The structure of a market is defined by the number of legitimate sellers of a particular product and the nature of competition among them. The two simplest forms of market are monopoly and duopoly.

Monopoly, as its Greek etymology indicates (mónos means one and pólein, to sell), is a structure reached by legal privilege or other agreements in which a single seller gains exclusive rights over a specific good or service. Because of the lack of competition, a monopoly allows firms to control the market, set prices and consequently hurt consumers when ill regulated. Common examples of monopoly can be found in the utilities market of most countries. Providers of water, electricity and natural gas are often granted exclusive rights to service municipalities through local governments.

Duopoly refers to a situation in which two sellers control all or nearly all of the market. Duopoly is the most basic type of oligopoly, a structure where few firms concentrate the majority of the market share. Classic examples of duopoly are Airbus and Boeing in the jet airliner market, DC Comics and Marvel in the superhero genre, Nvidia and AMD in PC GPU (Graphics Processing Unit) manufacturing. Netflix, that initially held the monopoly in streaming services, now faces competition with Amazon Prime.

Advertising competition is both dynamic and interactive among close rivals. A case in point is the Cola Wars, a term coined during Cold War to describe the long-running struggle between the two biggest brands in the soft drink industry: Coca-Cola Company and PepsiCo. In 1971, Coca-Cola experienced a massive increase in popularity after releasing its famous "I'd Like to Buy the World a Coke" jingle (in the most expensive commercial at the time). To regain ground, Pepsi started four

years later its widely known "Pepsi Challenge", an ongoing marketing campaign where consumers are encouraged to taste both sodas and blindly select which one they prefer (with test results leaning towards Pepsi as the consensus pick). This just being one example of the many back and forth exchanges between these two firms, that together account for 75% of the U.S. market [5], over the years. Since then, Coca-Cola and Pepsi continued to invest heavily on advertising in order to set themselves apart and dominate the soft drink market.

Optimal control theory has been employed in the theoretical studies of advertising models for decades. A recurrent topic of interest has been finding or characterizing an optimal advertising strategy over a defined period of time. As media vehicles evolve and information about consumers becomes more accessible to firms, studies of dynamic models in advertising have been growing in relevance.

More specifically, optimal control of Vidale-Wolfe [6] and related advertising dynamics has been studied in the context of obtaining analytical solutions in [4, 7–9]. Research has also focused on obtaining numerical solutions through the parameterization of the control action [10–13]. It is also worth mentioning, as ERICKSON [5] pointed out, that most of the work done has contemplated dynamic models operating in monopolistic markets, which often ignores important aspects of the marketing environment such as competition.

The main purpose of this work is to use modern software tools to find numerical solutions for the Vidale-Wolfe-Deal [7] and Three Populations duopoly models. Market dynamics are studied considering two distinct environments: co-operative and competitive. In the first case, the objective is to maximize the net gain of two firms simultaneously – which could be understood as a form of collusion. In the latter, these two firms compete with each other in a sequential game, based on a leader-follower iteration, each with the individual goal of maximizing profit on its turn. Simulations were conducted using the Open Source software JModelica.org [14] and JuMP [15] (Julia for Mathematical Optimization).

1.1 Motivation and objectives

Many markets which provide essential services and/or products and are not strictly duopolies do however function as such. Most of the studies regarding market dynamics choose to exclude the potential collusion aspect in duopoly markets from their analysis. When analyzing the competition between two firms, simulations are seldom done sequentially, which means that the capability of firm to adapt and react to its rival's strategy is not being considered.

Furthermore, despite this digital era of smartphones, Facebook and Instagram, most of the existing literature does not account for the effects of Word-of-Mouth

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(WOM) – more precisely electronic Word-of-Mouth (eWOM) – on sales-advertising models. This dissertation introduces a duopolistic version of the Three Population model, which takes the effects of eWOM in consideration.

1.1.1 Objectives

The objectives of this dissertation are:

- To verify the Vidale-Wolfe monopoly model's optimal response in order to validate subsequent results.

- To study the Vidale-Wolfe-Deal duopoly model under the assumption of competition and co-operation.

- To study the effects of eWOM and to propose a duopoly model based on the Three Populations model for monopolies.

1.2 Structure of the dissertation

The dissertation is organized as follows. Chapter 1 starts by giving a brief introduction to market dynamics, more specifically monopoly and duopoly models. It contextualizes the subject of the dissertation by providing recent examples of both practices in business. This also serves the purpose of highlighting the relevance of studying sales-advertising models.

In Chapter 2, the Vidale-Wolfe advertising model is presented along with Deal's extended version for a duopoly. The Chapter also offers the stability analysis of the Vidale-Wolfe-Deal model and recaps the Three Populations model. Dynamic optimization problems for all these advertising models are formulated.

Chapter 3 presents the methods and software used for solving the optimal control problems, also providing a brief overview of numerical methods. The dynamic optimization problems presented in the previous chapter are discretized and a Stackelberg competition based on the Vidale-Wolfe-Deal model is described.

In Chapter 4, numerical results for the Vidale-Wolfe monopoly model are discussed and compared to the theoretical solution found in [4]. The effects of eWOM on the Three Populations model are examined. Duopoly analysis is divided in two cases: co-operation and competition. The former discusses the obtained results for simultaneous optimization of the Vidale-Wolfe-Deal model, while the latter examines the proposed sequential Stackelberg competition.

Finally, Chapter 5 concludes the dissertation with a few remarks, summarizing the results and indicating some possible future work.

Chapter 2

Models for market share dynamics and associated optimal control problems: old and new

2.1 Vidale-Wolfe model

The model developed by VIDALE and WOLFE [6] considers two main aspects regarding the relation between sales and advertising. The rate of sales decreases with time if no advertisements are made, since consumers tend to forget about the product, which is modeled by an exponential decay term λS in (2.1). Advertising effort $u(t)$ results in a proportional increase in sales, modulated by diminishing returns as a direct consequence of marketing saturation, resulting in the term $\alpha u(t)\left(1 - \frac{S(t)}{M}\right)$ in (2.1). Hence, the Vidale-Wolfe model for a monopolistic firm can be stated as:

$$\dot{S}(t) = \alpha u(t)\left(1 - \frac{S(t)}{M}\right) - \lambda S(t) \tag{2.1}$$

where $S(t)$ is the rate of sales at time t, $u(t)$ is the advertising effort at the same instant, α is the response constant to the advertising effort, λ is the sales exponential decay constant and M is the market saturation – it is possible write M as a function of time, but in this dissertation, in common with most of the literature, M is chosen to be time-invariant.

A more convenient way to write equation (2.1) is to express the model in terms of the market share, which represents the rate of sales as a fraction of the market saturation. Thus, making the change of variable $(x(t) = S(t)/M)$ yields:

$$\dot{x}(t) = ru(t)(1 - x(t)) - \lambda x(t) \tag{2.2}$$

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where:

$$r = \frac{\alpha}{M}$$

DEAL [7] proposes and numerically analyses a version of the Vidale-Wolfe model for duopoly. In his model, each competitor is defined by its own particular sales response and sales decay constants. However, rival advertising has no assumed effect on the other firm's market share, only affecting the untapped portion of the market.

Similar to what was done in equation (2.2), Deal's model for duopoly can be normalized into the following pair of differential equations:

$$\dot{x}_i(t) = r_i u_i(t)(1 - x_1(t) - x_2(t)) - \lambda_i x_i(t); \quad i = 1, 2. \tag{2.3}$$

where subscript i indexes the ith firm.

Normalization imposes constraints on the state space. For instance, any firm must have a nonnegative market shares at all times. Furthermore, the sum of all firm's market share must never surpass unity.

The state space constraints for the Vidale-Wolfe-Deal model are summarized as follows:

$$x_1(t) \geq 0 \tag{2.4}$$

$$x_2(t) \geq 0 \tag{2.5}$$

$$x_1(t) + x_2(t) \leq 1 \tag{2.6}$$

2.2 Stability analysis of the Vidale-Wolfe-Deal model

Since almost all the literature on the Vidale-Wolfe-Deal model has concentrated on optimal control problems, it has not been subjected to the standard stability analysis, which is an important first step in understanding any dynamical system.

In this section, we study the stability of the dynamic system described by the differential equations shown in (2.3) when subjected to a constant input, $\bar{u}_i$. The dynamic system is written as follows:

$$\begin{cases} \dot{x}_1 = r_1 \bar{u}_1(1 - x_1 - x_2) - \lambda_1 x_1 \\ \dot{x}_2 = r_2 \bar{u}_2(1 - x_1 - x_2) - \lambda_2 x_2 \end{cases}$$

where r_1, r_2, $\bar{u}_1$, $\bar{u}_2$, λ_1 and λ_2 are all positive real values.

Grouping terms in x_1 and x_2 on the right hand side yields:

$$\begin{cases} \dot{x}_1 = -(r_1\bar{u}_1 + \lambda_1)x_1 - (r_1\bar{u}_1)x_2 + r_1\bar{u}_1 \\ \dot{x}_2 = -(r_2\bar{u}_2)x_1 - (r_2\bar{u}_2 + \lambda_2)x_2 + r_2\bar{u}_2 \end{cases} \tag{2.7}$$

Since the advertising efforts, $\bar{u}_1$ and $\bar{u}_2$, were assumed to be positive constants, it is easy to see that the Vidale-Wolfe-Deal model behaves as a linear system.

An equilibrium point is a state of the system that once reached does not change with time. Consequently, all state variables' derivatives must be equal to zero at any equilibrium point. Solving the dynamic system (2.7) when $\dot{x}_1 = \dot{x}_2 = 0$ yields:

$$x_1 = \frac{r_1\bar{u}_1\lambda_2}{(r_1\bar{u}_1\lambda_2 + r_2\bar{u}_2\lambda_1 + \lambda_1\lambda_2)} \tag{2.8}$$

$$x_2 = \frac{r_2\bar{u}_2\lambda_1}{(r_1\bar{u}_1\lambda_2 + r_2\bar{u}_2\lambda_1 + \lambda_1\lambda_2)} \tag{2.9}$$

Given that r_1, r_2, $\bar{u}_1$, $\bar{u}_2$, λ_1 and λ_2 were assumed to be positive real numbers, it is easy to verify that the solution shown in equations (2.8) and (2.9) ensures $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 < 1$.

This result shows that the equilibrium point for the Vidale-Wolfe-Deal model lies within the state space delimited by the constraints stated in equations (2.4 - 2.6).

To verify whether the equilibrium point given by equations (2.8) and (2.9) is stable, let $\mathcal{J}$ denote the Jacobian matrix of the dynamic system (2.7), which can be expressed as:

$$\mathcal{J} = \begin{bmatrix} -(r_1\bar{u}_1 + \lambda_1) & -(r_1\bar{u}_1) \\ -(r_2\bar{u}_2) & -(r_2\bar{u}_2 + \lambda_2) \end{bmatrix}$$

Consequently its determinant, $det(\mathcal{J})$, and trace, $tr(\mathcal{J})$, are given by:

$$det(\mathcal{J}) = r_1\bar{u}_1\lambda_2 + r_2\bar{u}_2\lambda_1 + \lambda_1\lambda_2 \quad >0$$
$$tr(\mathcal{J}) = -(r_1\bar{u}_1 + r_2\bar{u}_2 + \lambda_1 + \lambda_2) < 0$$

and this pattern of signs guarantees that the real part of its eigenvalues is strictly negative. Therefore, the equilibrium point of the dynamic system is locally asymptotically stable.

Furthermore, the discriminant of the characteristic polynomial of second degree can be written as:

$$tr^2(\mathcal{J}) - 4det(\mathcal{J}) = [(r_1\bar{u}_1 + \lambda_1) + (r_2\bar{u}_2 + \lambda_2)]^2 - 4[(r_1\bar{u}_1 + \lambda_1)(r_2\bar{u}_2 + \lambda_2) - (r_1\bar{u}_2)(r_2\bar{u}_1)]$$

$$= (r_1\bar{u}_1 + r_2\bar{u}_2)^2 + (\lambda_1 - \lambda_2)^2 + 2(\lambda_1 - \lambda_2)(r_1\bar{u}_1 - r_2\bar{u}_2)$$

For the particular case where both firms have the same sales decay constant, i.e. $\lambda_1 = \lambda_2$, we have:

$$tr^2(\mathcal{J}) - 4det(\mathcal{J}) = (r_1\bar{u}_1 + r_2\bar{u}_2)^2 > 0$$

Thus, both eigenvalues are distinct negative real numbers and the critical point is a stable node.

For $\lambda_1 \neq \lambda_2$, the critical point can be either a stable node, if $tr^2(\mathcal{J}) - 4det(\mathcal{J}) \geq 0$, or a stable focus, $tr^2(\mathcal{J}) - 4det(\mathcal{J}) < 0$. Control literature may sometimes refer to the critical point as a star rather than a node when the discriminant is equal to zero. Regardless, the critical point is clearly an equilibrium point.

Therefore, not only an equilibrium point exists, it is also locally asymptotically stable and lies in the nonnnegative quadrant,i.e., is feasible.

As a conclusion of the stability analysis of the Vidale-Wolfe-Deal model, any control that attains final constant positive values leads to stable equilibrium of market shares.

2.3 Optimal control problem for the Vidale-Wolfe-Deal model

In this section we formulate an optimal control model associated to the Vidale-Wolfe-Deal model. It is assumed that the objective of the company is to maximize the net gain, i.e. profit accrued from market share (=state) minus the advertising expense (=control), for a defined period of time, T_f. Thus, the cost function can be formulated as:

$$J_i(x_i, u_i, T_f) = \int_0^{T_f} c_i x_i(t) - u_i(t)dt; \quad i = 1, 2. \tag{2.10}$$

where c_i denotes the maximum revenue potential of a particular firm, which assumes a constant margin per unit product.

In a cooperative duopoly, both firms have the objective of maximizing their individual net gains. It is possible to introduce a nonnegative weight μ such that if $\mu = 1$, then both gains are weighted equally. If not, then one firm is favored over the other. This weighted cost function can be written as:

$$J_{12}(x_1, x_2, u_1, u_2, T_f) = J_1(x_1, u_1, T_f) + \mu J_2(x_2, u_2, T_f) \tag{2.11}$$

The optimal control problem can be summarized as follows:

$$\underset{u_1, u_2}{\text{maximize}} \quad J_{12}(x_1, x_2, u_1, u_2, T_f) \tag{2.12a}$$

$$\text{subject to} \quad \dot{x}_i(t) = r_i u_i(t)(1 - x_1(t) - x_2(t)) - \lambda_i x_i(t); \quad i = 1, 2, \tag{2.12b}$$

$$u_i(t) \leq u_{i_{max}} \qquad i = 1, 2, \tag{2.12c}$$

$$u_i(t) \geq 0 \qquad i = 1, 2, \tag{2.12d}$$

$$x_i(t) \geq 0 \qquad i = 1, 2, \tag{2.12e}$$

$$x_1(t) + x_2(t) \leq 1 \tag{2.12f}$$

where equation (2.12b) expresses the dynamics, equations (2.12c - 2.12d) are constraints on the controls, with $u_{i_{max}}$ being the maximum budget available for advertising to the ith company, and equations (2.12e - 2.12f) are the normalization constraints on the state space.

Rewriting the problem for a monopolistic firm is straightforward, as equations (2.13a - 2.13f) show. It only requires replacing equation (2.12b) with (2.2) and modifying the cost function and its constraints accordingly.

$$\underset{u}{\text{maximize}} \quad J(x, u, T_f) = \int_0^{T_f} cx(t) - u(t)dt \tag{2.13a}$$

$$\text{subject to} \quad \dot{x}(t) = ru(t)(1 - x(t)) - \lambda x(t), \tag{2.13b}$$

$$x(t) \leq 1, \tag{2.13c}$$

$$x(t) \geq 0, \tag{2.13d}$$

$$u(t) \leq u_{max}, \tag{2.13e}$$

$$u(t) \geq 0 \tag{2.13f}$$

2.4 Three Populations model

This section proposes a duopolistic version of the model proposed by BHAYA and KASZKUREWICZ [1]. Their model is "inspired by Lotka-Volterra type models of three interacting populations of customers: one that is satisfied with the brand that the model seeks to describe, its market share at time t (fraction of the total costumer population) being denoted $x_b(t)$, the fraction of defectors from the brand, denoted $x_d(t)$, and the fraction of undecided customers being denoted $x_i(t)$" [1]. In accordance with the terminology in [1], the three populations are referred to brand, defectors and undecided respectively. In addition, the corresponding normalized market share of each costumer population must sum to unity at all times, i.e.:

$$x_b(t) + x_d(t) + x_i(t) = 1 \qquad (2.14)$$

This model assumes that the advertising effort u made by the brand increases its market share by depleting the undecided and, possibly, defector populations. Moreover, the undecided fraction of costumers is treated as prey and is, thus, subjected to predation by both brand and defectors. As in Lotka-Volterra type models, all predation terms are proportional to the encounters between any two populations, and this assumption is also similar to the one made made in word-of-mouth (WOM) or electronic word-of-mouth (eWOM) models [16], and also in competitive dynamics of web sites [17].

The system describing the Three Populations model is expressed by the following set of equations:

$$\begin{cases} \dot{x}_b = k_b u + k_{bi} x_b x_i - k_{bd} x_b x_d \\ \dot{x}_d = -k_d u + k_{di} x_d x_i + k_{bd} x_b x_d \\ \dot{x}_i = (k_d - k_b) u - k_{di} x_d x_i - k_{bi} x_b x_i \end{cases} \qquad (2.15)$$

with u denoting positive advertising effort by the brand and parameters k_b, k_d, k_{bd}, k_{bi}, k_{di} expressing the rates at which populations grow or diminish.
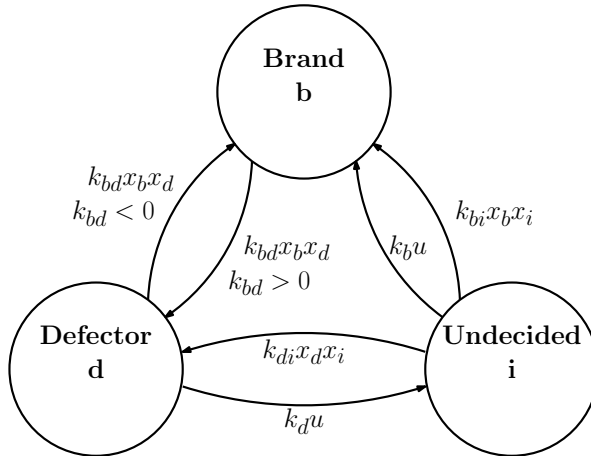
Figure 2.1: Graph showing interactions among brand, defector and undecided populations (nodes). Arrows are labeled with a term that indicates the rate at which the population at the tail is depleted, and, correspondingly, the population at the head of the arrow is growing. Figure based on [1].

The interaction graph among the three populations is shown in Figure 2.1. Advertising is assumed to have a positive effect on the brand by swaying undecided costumers and, possibly, depleting the defector population. Hence, constants k_b and k_d are respectively positive and nonnegative. Because the undecided population is modeled as prey, constants k_{bi} and k_{di} can only take positive values. The sign

of k_{bd} determines the flow between brand and defectors after each encounter. If $k_{bd} = 0$, it is being assumed that interactions have no effect on the growth rates of the respective populations.

An inspection of the equations describing the dynamic system (2.15) shows that the sum of all population's derivatives (size variations) is zero, which implies that the total customers population is time-invariant. Therefore, as long as the initial conditions are chosen such that $x_b(0) + x_d(0) + x_i(0) = 1$, constraint (2.14) holds true for all time.

We now propose a duopolistic version of the Three Populations model by replacing the defectors with a second rival brand, hereafter referred to rival and denoted $x_r(t)$. Competition is introduced by allowing the rival to counter the advertising effort of the brand with an effort of its own, denoted $u_r(t)$. Furthermore, it is assumed that encounters between rival and brand result in clients of both brands becoming undecided, which depletes both brand and rival populations and lead to corresponding increases in the undecided market share. The Three Populations model for duopolies is written as follows:

$$
\begin{cases}
\dot{x}_b = k_b u_b + k_{bi} x_b x_i - \alpha k_{br} x_b x_r \\
\dot{x}_r = k_r u_r + k_{ri} x_r x_i - (1 - \alpha) k_{br} x_b x_r \\
\dot{x}_i = -k_b u_b - k_r u_r - k_{bi} x_b x_i - k_{ri} x_r x_i + k_{br} x_b x_r
\end{cases}
\tag{2.16}
$$

where parameter k_r is a positive growth constant similar to k_b and α is a parameter between 0 and 1 that determines the proportion in which brand and rival populations are depleted. For $\alpha = 0.5$, both decrease equally on each encounter. Figure 2.2 shows the new interaction graph among the three populations.

As in the previous model, the sum of the differential equations describing the dynamic system (2.16) is also zero. Thus, the modified Three Populations model still satisfies the basic invariance assumption, i.e., the non-violation of constraint (2.14).

The invariance constraint, however, does not guarantee that the market shares of brand, rival and undecided populations always remain nonnegative. In order to avoid any market share taking negative values, there is still need to enforce the set of non-negativity constraints:

$$x_b(t) \geq 0 \tag{2.17}$$

$$x_r(t) \geq 0 \tag{2.18}$$

$$x_i(t) \geq 0 \tag{2.19}$$
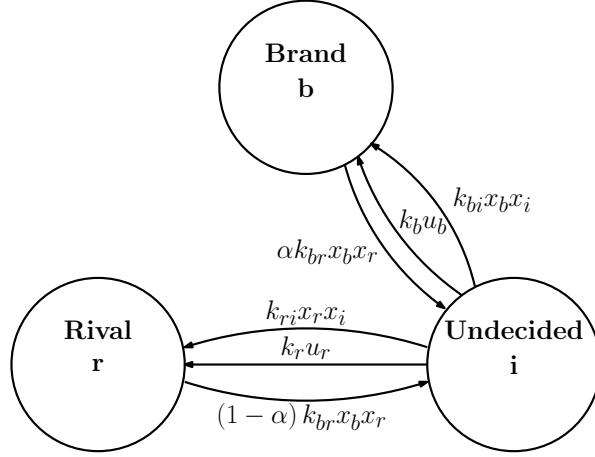
Figure 2.2: Graph showing interactions among brand, rival and undecided populations (nodes) for the duopoly model. Arrows are labeled with a term that indicates the rate at which the population at the tail is depleted, and, correspondingly, the population at the head of the arrow is growing. The duopolistic model assumes that all interactions between brand and rival lead to an increase of the undecided population. Corresponding decreases to the brand and rival populations are modeled according to a parameter α that determines the proportion in which each population is depleted.

Similarly to what was done in Section 2.3, the optimal control problem associated with the duopolistic version of the Three Populations model can be written as:

$$\underset{u_b, \, u_r}{\text{maximize}} \quad J_{br}(x_b, x_r, u_b, u_r, T_f) \tag{2.20a}$$

$$\text{subject to} \quad \dot{x}_b = k_b u_b + k_{bi} x_b x_i - \alpha k_{br} x_b x_r, \tag{2.20b}$$

$$\dot{x}_r = k_r u_r + k_{ri} x_r x_i - (1 - \alpha) k_{br} x_b x_d, \tag{2.20c}$$

$$\dot{x}_i = -k_b u_b - k_d u_d - k_{di} x_d x_i - k_{bi} x_b x_i + k_{br} x_b x_r, \tag{2.20d}$$

$$x_b \geq 0, \tag{2.20e}$$

$$x_r \geq 0, \tag{2.20f}$$

$$x_i \geq 0, \tag{2.20g}$$

$$u_b \leq u_{b_{max}}, \tag{2.20h}$$

$$u_b \geq 0, \tag{2.20i}$$

$$u_r \leq u_{r_{max}}, \tag{2.20j}$$

$$u_r \geq 0 \tag{2.20k}$$

where $u_{b_{max}}$ and $u_{d_{max}}$ are the maximum advertising effort achievable for brand and rival populations respectively. For brevity, the time arguments of all advertising and market share were omitted.

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The cost function in equation (2.20a) was chosen as:

$$J_{br}(x_b, x_r, u_b, u_r, T_f) = \int_0^{T_f} [c_b x_b(t) - u_b(t) + c_r x_r(t) - u_r(t)]\, dt,$$

which is equivalent to the unweighted cost function adopted for simultaneous competition in equation (2.11).

Chapter 3

Mathematical formulations and numerical solution methods for optimal control problems of market share dynamics

In this chapter, we give a brief overview on the use of software JModelica.org [14] and JuMP [15] to solve the dynamic optimization problem presented in the last section of chapter 2. A sequential Game based on a Leader-Follower approach for the Duopoly Competition using the Vidale-Wolfe-Deal advertising model is also discussed.

3.1 Solving dynamic optimization problems with JModelica.org

JModelica.org is a tool targeting model-based analysis of large-scale dynamic systems, in particular dynamic optimization [2]. It uses the modeling language Modelica to describe system dynamics, and Optimica, a Modelica language extension, to formulate the optimization problem. Optimica allows the formulation of a continuous-time optimal control problem in its natural form, with the tool handling the details of the discretization in a manner that is transparent to the user, which can be rather complex. The user's job, thus, is considerably simplified.

3.1.1 Numerical methods for optimal control problems

In optimal control theory, there are many approaches to numerically solve optimization problems. The earliest numerical methods date back to the late 50s and stem from the works of Bellman [18–21] in Dynamic Programming. Since then, the com-

plexity and variety of optimal control applications has vastly increased. Nowadays, the most widely used numerical methods for solving optimal control problems are based on first-order necessary conditions for local optimality, and fall into two major categories: direct and indirect methods.

Indirect methods start by establishing the optimality conditions. The ensuing differential equations are then discretized and a numerical solution is found. The optimality conditions are based on calculus of variations and Pontryagin's maximum principle [22, 23], and appear in the form of boundary value problems. The solution of a boundary value problem is found by solving a system of difference equations that satisfies endpoint and/or interior point conditions.

In an alternative approach, direct methods first discretize the dynamics and then establish the optimality conditions. They reduce an infinite-dimensional problem (continuous state space) to a finite-dimensional one (discrete state space) by transcribing the original optimal control problem into a nonlinear programming problem (NLP). The optimality conditions for the NLP are then given by the Karush-Kuhn-Tucker (KKT) conditions.

Regardless of the approach, both direct and indirect methods require the solution of difference equations in order to solve dynamic optimization problems. Therefore, it is not uncommon to find the same numerical techniques being employed in both approaches. An overview of numerical methods, based on [2], is presented in Figure 3.1.
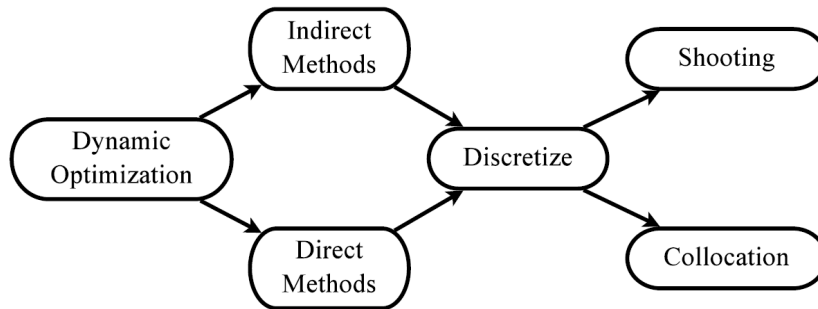
Figure 3.1: Overview of decomposition of a continuous-time dynamic optimization problem, in the direct and indirect approaches, into discrete-time problems, solvable by the numerical methods denominated shooting and collocation. Figure from [2], used with permission.

Shooting and collocation are two of the most widespread methods often used in direct and indirect approaches. The simplest form of shooting is called single shooting and consists of making an initial guess of either the control parameters (direct method) or the unknown boundary conditions at one end of the interval (indirect method), and then integrating the resulting IVP (Initial Value Problem) along the time horizon. If the specified conditions at the other end are not attained,

the initial guess is adjusted and the process is repeated. The name of the method comes from the fact that it can be understood as basic feedback control employed to set the angle of a cannon in order to hit a target [24]. If the target is missed, the angle is adjusted based on the previous shot and the cannon is fired again.

Single shooting is an appealing method due to its simplicity, but can present significant numerical difficulties since it is highly sensitive to the initial guess, propagating its error as time marches. This may cause instability even when the boundary value problem itself is well conditioned. The numerical robustness of single shooting can be improved by dividing the time horizon into several subintervals. This method is called multiple shooting, and essentially decouples the dynamics by introducing interior point boundary values as variables and imposing linking constraints between adjacent subintervals. Single shooting is then applied within each new subinterval. Despite the increased size of the problem due to the extra variables, multiple-shooting is an improvement over single shooting because integration is performed over a significantly smaller time period, thus reducing the sensitivity to errors in the initial guess. Nevertheless, even multiple shooting can present issues unless a sufficiently good guess is given.

Collocation methods, on the other hand, consist on choosing a number of points in the domain, called collocation points, and fitting a polynomial solution, up to a chosen degree, while satisfying the imposed constraints at each collocation point. There are different ways to choose collocation points, each resulting in a particular stability and order of convergence. The most common ones are based on Gauss, Radau and Lobatto quadratures [25].

In an indirect collocation method, state and adjoint variables are parameterized using piecewise polynomials. The collocation procedure leads to a root-finding problem where the dynamic constraints can be written as an algebraic vector of the coefficients of the piecewise polynomial. This system of nonlinear equations is then solved using an appropriate root-finding technique. The region of convergence of indirect methods tends to be quite narrow, thus requiring good initial guesses, including guesses of the adjoint functions. When a problem has inequality path constraints, a priori estimates of the sequence of constrained arcs are needed, which may be hard to find[3].

In a direct collocation method, the differential equations are discretized by defining a grid of N collocation points covering the time interval $[t_0, t_f]$ and the resulting difference equations become a finite set of equality constraints of the NLP problem. Figure 3.2 [3] illustrates the idea of a discretized control u and state x in the interval $t \in [t_0, t_f]$, with $t_0 < t_1 < t_2 < \cdots < t_N = t_f$. The NLP problems that arise from direct collocation can be very large, having possibly hundreds of thousands of variables and constraints. However, they are usually quite sparse, making them easier

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to solve than boundary value problems. Moreover, there is no need to explicitly derive the necessary conditions of the continuous problem, which is more attractive in complex cases, and they do not require an a priori specification of the sequence of constrained arcs in problems with inequality constraints [3].
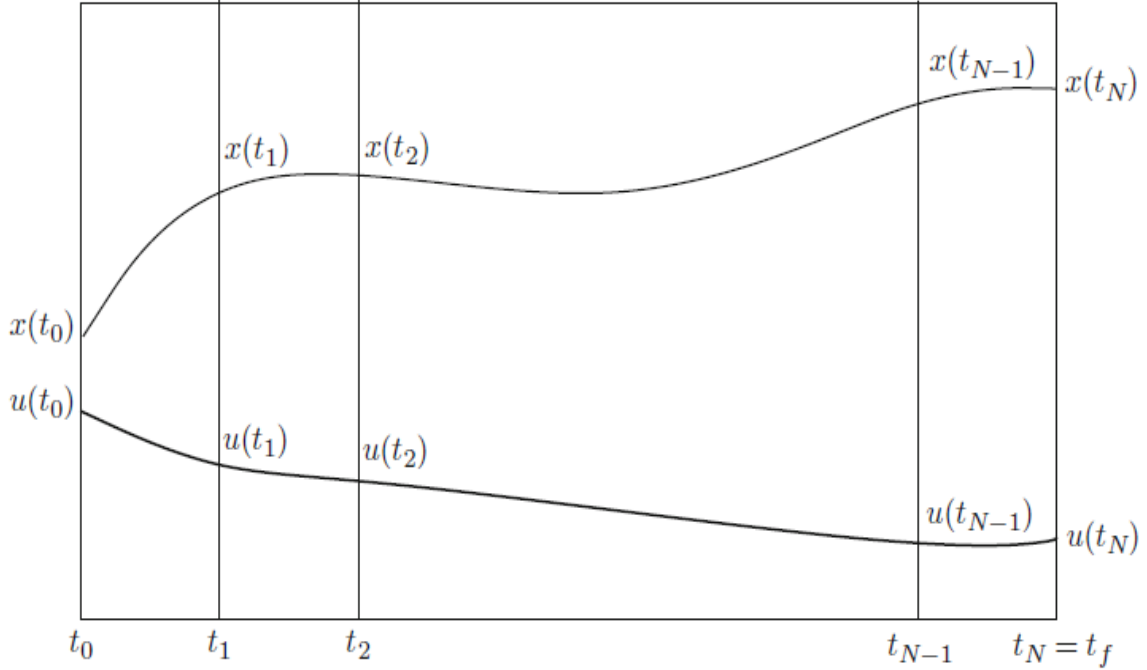
Figure 3.2: Control u and state x discretized in the interval $t \in [t_0, t_f]$. Figure from [3], used with permission (please see Appendix C).

When using Runge-Kutta methods for discretization, collocation is said to simultaneously solve differential equations because all the unknown parameters are determined at the same time. Furthermore, collocation methods simulate the dynamics of the system implicitly because the values of the state at each collocation point are obtained simultaneously rather than sequentially [24].

Collocation methods can be either local, where the time horizon is divided into subintervals and low-order polynomials are used to approximate the trajectories within each time frame, or global, where a single high-order polynomial is used over the entire time horizon.

JModelica.org uses a method based on direct local collocation, with support for Gauss and Radau points, to transcribe the problem into a NLP. A local optimum to the NLP is then found by solving the first-order KKT conditions, using iterative techniques based on Newton's method [2]. JModelica.org uses CasADi [26] (Computer algebra system with Automatic Differentiation) in order to obtain first and second-order derivatives of the NLP cost and constraint functions with respect to its variables. CasADi offers interfaces to third-party numerical optimization solvers

such as IPOPT [27].

Figure 3.3 shows the compilation process in JModelica.org. It starts with the user-provided Modelica and Optimica code. The Modelica model is then flattened in order to get a representation that is closer to a differential algebraic equation (DAE) system. The flat representation essentially consists of only variable declarations and equations. The compilation process ends with a symbolic representation of the NLP in CasADi [2].
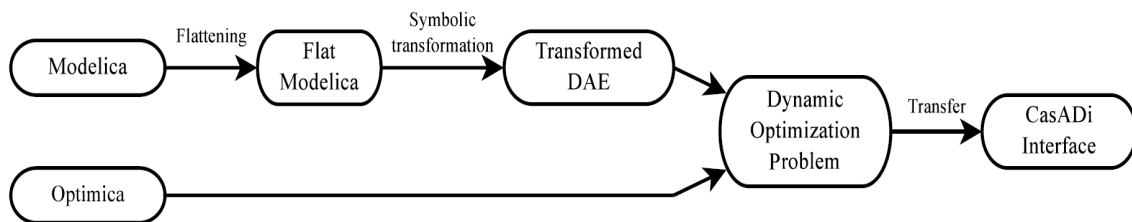
Figure 3.3: Compilation Process in JModelica.org for Dynamic Optimization Problems. Figure from [2], used with permission

3.2 Solving dynamic optimization problems with JuMP

JuMP is an AML (Algebraic Modeling Language) – a computer programming language that allows users to express a wide range of optimization problems (linear, mixed-integer, quadratic, conic-quadratic, semidefinite, and nonlinear) in a high-level algebraic syntax – embedded in Julia [28], which is a general purpose high level language for scientific computation. It provides not only an efficient open-source alternative to commercial systems but also takes advantage of a number of features of Julia which are unique among programming languages for scientific computing [15].

For instance, in most languages other than Julia, the most common approach to capture user's input is operator overloading. Essentially, it extends the language's definition of basic arithmetical operators to build data structures representing expressions. Unfortunately, this method leads to an increase in complexity as it needs to store, for example, constants, coefficient vectors, index sets and decision variables in order to build math like statements. As a result, simple operations like addition and subtraction are no longer fast, constant-time operations; a property which is usually overlooked in the case of floating-point numbers. JuMP, on the other hand, does not rely on operator overloading when capturing a user's input, instead turning to an advanced feature of Julia called syntactic macros.

As in Lisp, the input of a macro is a data structure of the language itself, not just a string of text. By defining variables, constraints and objective as macros, JuMP provides a natural syntax for algebraic modeling without the need of a custom text-based parser and drawbacks related to operator overloading. Moreover, JuMP is able to efficiently process large-scale problems by exploiting well-known structural properties since the code is represented by objects that can be created and manipulated from within the language

JuMP uses techniques from automatic differentiation (AD) to evaluate derivatives of user-defined expressions and is designed to be extensible, allowing for developers both to plug in new solvers, such as IPOPT, for existing problem classes and to extend the syntax of JuMP itself to new classes of problems. This also allows users to test the efficiency of different solvers for a specific problem, without the need to rewrite the whole code.

Due primarily to the compilation time, JuMP has a noticeable start-up cost of a few seconds even for the smallest instances. However, if a family of models is solved multiple times within a single session, this cost of compilation is only paid for the first time that an instance is solved. Therefore, when solving a sequence of instances in a loop, the amortized cost of compilation is negligible. This is a particularly attractive feature considering the Stackelberg competition described in Section 3.3 will essentially require solving two different models that differ only on the right-hand side coefficients of the constraints at each iteration.

3.2.1 Discretization of the Vidale-Wolfe-Deal advertising model

Unlike JModelica.org, that allows the optimal control problem to be formulated in its continuous form, JuMP requires the formulation to be discretized.

Discretization of the optimal control problem for both monopoly and duopoly models, presented in section 2.3, is a fairly straightforward procedure that can be done using different techniques. In this work we used the Forward Euler Method, that adopts the following approximation:

$$\dot{y} = \frac{dy}{dt} \approx \frac{y(k+1) - y(k)}{\Delta t} \tag{3.1}$$

Applying the approximation shown in (3.1) to the equation (2.3) results in the discretized version of equation (2.12b):

$$x_i(k+1) = x_i(k) + f_{i,k}(x_1, x_2, u_i); \quad i = 1, 2.$$

where:

$$f_{i,k}(x_1, x_2, u_i) = \Delta t[r_i u_i(k)(1 - x_1(k) - x_2(k)) - \lambda_i x_i(k)].$$

Discretization of the net gain cost function, equation (2.10), can be achieved by simply approximating the integration to the sum of the net gain at all N intervals as shown:

$$J_i(x_i, u_i, N) = \sum_{k=1}^{N} (c_i x_i(k) - u_i(k)); \quad i = 1, 2.$$

Consequently equation (2.11) becomes:

$$J_{12}(x_1, x_2, u_1, u_2, N) = J_1(x_1, u_1, N) + \mu J_2(x_2, u_2, N)$$

Thus, the optimal control problem can be described by equations (2.12a - 2.12f) can be discretized as:

$$
\begin{align}
& \underset{u_1, u_2}{\text{maximize}} \quad J_{12}(x_1, x_2, u_1, u_2, N) && \text{(3.2a)} \\
& \text{subject to} \quad x_1(k+1) = x_1(k) + f_{1,k}(x_1, x_2, u_1); && k = 1, ..., N, && \text{(3.2b)} \\
& \qquad\qquad\quad x_2(k+1) = x_2(k) + f_{1,k}(x_1, x_2, u_2); && k = 1, ..., N, && \text{(3.2c)} \\
& \qquad\qquad\qquad x_1(k) + x_2(k) \leq 1 && k = 1, ..., N+1, && \text{(3.2d)} \\
& \qquad\qquad\qquad x_1(k) \geq 0; && k = 1, ..., N+1, && \text{(3.2e)} \\
& \qquad\qquad\qquad x_2(k) \geq 0; && k = 1, ..., N+1, && \text{(3.2f)} \\
& \qquad\qquad\qquad u_1(k) \leq u_{1_{max}}; && k = 1, ..., N, && \text{(3.2g)} \\
& \qquad\qquad\qquad u_1(k) \geq 0; && k = 1, ..., N, && \text{(3.2h)} \\
& \qquad\qquad\qquad u_2(k) \leq u_{2_{max}}; && k = 1, ..., N, && \text{(3.2i)} \\
& \qquad\qquad\qquad u_2(k) \geq 0; && k = 1, ..., N && \text{(3.2j)}
\end{align}
$$

The formulation (3.2a - 3.2j) is easy to understand and natural; however, it is not in the standard form acceptable as input to most mathematical programming solvers. When using a solver for such problems, the latter should be rewritten in standard NLP form [29], i. e., writing all constraints for all variables at all instants. For this particular and relatively small discrete NLP, it means writing all $9N + 3$ constraints explicitly. This can become quite a tedious task for problems with a large number of variables or with small sampling time, even when taking advantage of block matrix notation as shown below.

The corresponding NLP can be written compactly in matrix form as follows:

$$\underset{z}{\text{maximize}} \quad c^T z$$

$$\text{subject to} \quad r(z) = \bar{0},$$

$$Az \leq \mathcal{U},$$

$$Az \geq \mathcal{L}$$

where:

$$
c = \begin{bmatrix} c_1 \\ \vdots \\ c_1 \\ 0 \\ c_2 \\ \vdots \\ c_2 \\ 0 \\ -1 \\ \vdots \\ -1 \end{bmatrix}
\qquad
z = \begin{bmatrix} x_1(1) \\ \vdots \\ x_1(N) \\ x_1(N+1) \\ x_2(1) \\ \vdots \\ x_2(N) \\ x_2(N+1) \\ u_1(1) \\ \vdots \\ u_1(N) \\ u_2(1) \\ \vdots \\ u_2(N) \end{bmatrix}
$$

$$
r(z) = \begin{bmatrix} x_1(2) - x_1(1) - f_{1,1}(x_1, x_2, u_1) \\ \vdots \\ x_1(N+1) - x_1(N) - f_{1,N}(x_1, x_2, u_1) \\ x_2(2) - x_2(1) - f_{2,1}(x_1, x_2, u_2) \\ \vdots \\ x_2(N+1) - x_2(N) - f_{2,N}(x_1, x_2, u_2) \end{bmatrix}
$$

$$
\mathcal{L} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
\quad
A = \begin{bmatrix} I_{N+1} & I_{N+1} & & & \\ I_{N+1} & & & & \\ & & I_{N+1} & & \\ & & & I_N & \\ & & & & I_N \end{bmatrix}
\quad
\mathcal{U} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ u_{1_{max}}^N \\ u_{2_{max}}^N \end{bmatrix}
$$

such that A is a sparse matrix with all elements left in blank being equal to zero, I_N is the $N \times N$ identity matrix and $u_{i_{max}}^N$ is the $N \times 1$ vector with all elements equal to $u_{i_{max}}$.

In this context, the most attractive feature of JuMP is that it allows optimization problems to be formulated not only in its matrix form above, which can be advantageous for simpler problems, but also in its concise and more natural form (3.2a - 3.2j), which includes handling constraints inside a loop. In this work, we will transcribe all the optimization problems studied into standard form NLPs for the sake of completeness. However, all coding in Julia was done using natural formulation (3.2a - 3.2j).

Analogously, the optimization problem for a monopolistic firm described by equations (2.13a - 2.13f) can be discretized as:

$$\text{maximize}_{u} \quad J(x, u, N) = \sum_{k=1}^{N} (cx(k) - u(k)) \tag{3.3a}$$

$$\text{subject to} \quad x(k+1) = x(k) + f_k(x, u); \quad k = 1, ..., N, \tag{3.3b}$$

$$0 \leq x(k) \leq 1; \qquad k = 1, ..., N+1, \tag{3.3c}$$

$$0 \leq u(k) \leq u_{max}; \qquad k = 1, ..., N \tag{3.3d}$$

where:

$$f_k(x, u) = \Delta t[ru(k)(1 - x(k)) - \lambda x(k)],$$

wich can be transcribed into the following NLP:

$$\text{maximize}_{z} \quad c^T z$$

$$\text{subject to} \quad r(z) = \bar{0},$$

$$Az \leq \mathcal{U}_b,$$

$$Az \geq \mathcal{L}_b$$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_1 \\ 0 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \qquad z = \begin{bmatrix} x(1) \\ \vdots \\ x(N) \\ x(N+1) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix}$$

$$r(z) = \begin{bmatrix} x(2) - x(1) - f_1(x, u) \\ \vdots \\ x(N+1) - x(N) - f_N(x, u) \end{bmatrix}$$

$$\mathcal{L} = \mathbb{0}^{2N+1} \quad A = I_{2N+1} \quad \mathcal{U} = \begin{bmatrix} \mathbb{1}^{N+1} \\ u^N_{max} \end{bmatrix}$$

where $\mathbb{0}^N$ and $\mathbb{1}^N$ denote the $N \times 1$ vectors with all elements equal to 0 and 1, respectively.

Clearly, constraints (3.2b) and (3.2c) in the duopoly model and (3.3b) in the monopoly model are nonlinear. This knowledge leads to two questions: Is the feasible region convex? If the feasible region is non-convex, how does it affects the optimal solution?

To answer these questions we will analyze the simpler of the two models, the Vidale-Wolfe monopoly model.

Let us write equations (3.3a - 3.3d) for $N = 2$. For brevity, it is assumed $\Delta t = 1$. Cost function J in equation (3.2a) can be written as:

$$J = cx(2) + cx(1) - u(2) - u(1)$$

The dynamic constraint stated in equation (3.2b) becomes:

$$x(2) = x(1) + ru(1) - ru(1)x(1) - \lambda x(1)$$

Lastly, the box constraints given by equations (3.2c) and (3.2d) can be summarized as follows:

$$0 \le x(1) \le 1$$
$$0 \le x(2) \le 1$$
$$0 \le u(1) \le u_{max}$$
$$0 \le u(2) \le u_{max}$$

Since $u(2)$ has a negative effect on the cost function that we seek to maximize and it does not appear in the market share dynamics (as we limited $N = 2$ and $u(2)$ only affects the future state $x(3)$, which is not considered), its optimum value is obviously zero.

To simplify the analysis, let the optimization variables be renamed as:

$$x(1) = z_1, \quad x(2) = z_2, \quad u(1) = z_3, \quad u(2) = z_4 = 0$$

The optimization problem to be solved can then be written as:

$$\underset{z_1, z_2, z_3}{\text{maximize}} \quad cz_1 + cz_2 - z_3 \tag{3.4a}$$

$$\text{subject to} \quad z_2 - (1 - \lambda)z_1 + rz_1z_3 - rz_3 = 0, \tag{3.4b}$$

$$0 \le z_1 \le 1, \tag{3.4c}$$

$$0 \le z_2 \le 1, \tag{3.4d}$$

$$0 \le z_3 \le u_{max} \tag{3.4e}$$

Writing equation (3.4b) in its algebraic form yields:

$$z^T A z + b^T z = 0,$$

where:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \frac{1}{2}r \\ 0 & 0 & 0 \\ \frac{1}{2}r & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} \lambda - 1 \\ 1 \\ r \end{bmatrix}$$

Matrix A is symmetric with zeros on the principal diagonal. Since its principal diagonal is null, matrix A can, at best, be positive semi-definite. To show that this can not be the case, consider its sub-matrix:

$$\begin{bmatrix} 0 & \frac{1}{2}r \\ \frac{1}{2}r & 0 \end{bmatrix}$$

This sub-matrix has eigenvalues $\pm\frac{1}{2}r$, violating the necessary condition for positive semi-definiteness. This guarantees the indefiniteness of matrix A, which could imply non-convexity of the zero set (3.4b).

Examining further the dynamic constraint (3.4b), note that, since the optimization variable z_2 occurs by itself in the constraint, it can be chosen as a free parameter (called, say, $\sigma \in [0, 1]$) in order to investigate the relation between z_1 and z_3. This leads to:

$$\sigma - (1 - \lambda)z_1 + rz_1z_3 - rz_3 = 0$$

Rearranging z_1 as a function of z_3 and the parameter σ yields:

$$z_1 = \frac{rz_3 - \sigma}{rz_3 - (1 - \lambda)}$$

Renaming $rz_3 = \phi$, such that $\phi \in [0, r]$, and $(1 - \lambda) = \gamma$, such that $\gamma \in [0, 1]$, leads to the simplified expression:

$$z_1 = \frac{\phi - \sigma}{\phi - \gamma}$$

Since $z_1 \in [0, 1]$, if $\sigma < \gamma$, then $\phi \leq \sigma$. Otherwise, if $\sigma > \gamma$, then $\phi \geq \sigma$. Therefore, in summary:

$$\forall z_1 \in [0, 1] : z_2 < (1 - \lambda) \implies rz_3 \leq z_2$$

$$\forall z_1 \in [0, 1] : z_2 > (1 - \lambda) \implies rz_3 \geq z_2$$

Nonetheless, z_1 is the initial market share of a firm. Because it assumes a fixed value, the optimization variable can be replaced by a constant (called $\delta \in [0, 1]$). Because z_1 is fixed, the feasible region is given defined by the line:

$$z_2 = r(1 - \delta)z_3 + (1 - \lambda)\delta \tag{3.5}$$

delimited by the box constraints.

Substituting equation (3.5) in the cost function (3.4a) yields:

$$J = \alpha z_3 + \beta$$

where:

$$\alpha = cr(1 - \delta) - 1$$
$$\beta = c(2 - \lambda)\delta$$

For $\alpha \leq 0$, i.e $\delta \geq 1 - 1/cr$, the performance index J is maximum when z_3 is equal to zero. Otherwise, the maximum value of J occurs when z_3 is maximum. It is important to observe that the advertising effort z_3 not only is limited by the available budget (u_{max}) but it is also constrained by the dynamics, i.e it can not assume values that lead the final market share, z_2, to be greater than unity. For brevity, we refer to this threshold as u_{dyn}, which is given as follows:

$$u_{dyn} = \frac{1 - (1 - \lambda)\delta}{r(1 - \delta)}$$

Hence, for $\alpha > 0$, i.e $\delta < 1 - 1/cr$, the performance index J is maximum when z_3 is equal to $\min(u_{dyn}, u_{max})$.

Although a little tedious, this argument can be generalized to the case when $N > 2$. The general conclusion is that, for this particular problem, despite the possible non-convexity of the zero set defined by the equality constraints (dynamics), the analysis of the resulting mathematical programming problem is possible and there are a finite number of possibilities, determined by the initial conditions and parameters (c, r, λ) of the problem.

3.3 Sequential Game

This work proposes to study the sequential game based on leader-follower iteration for duopoly competition using Vidale-Wolfe-Deal advertising model. The game starts in a monopoly, with a single player, labeled Firm A, having an initial market share equal to zero. Firm A will then select an advertising strategy that maximizes its own net gain, ending the round. After the first round is over, Firm A becomes the leader with a known (public) advertising strategy, u_a^1, and a final market share, $x_a^1(N+1)$; where the notation $x\{u\}_i^j$ reads market share {advertising strategy} for firm i at round j. In the following round, a second player, labeled Firm B, joins the market, which now becomes a duopoly. Firm B, the follower, knows u_a^1 but starts with an initial market share, $x_b^2(1) = 0$. Firm A, the leader, does not know Firm B's advertising efforts, repeats its previous successful strategy, $u_a^2 = u_a^1$, and starts the round with its latest achieved market share, $x_a^2(1) = x_a^1(N+1)$. Firm B will then pick an advertising strategy, u_b^2, that maximizes its own net gain, taking into account Firm A's former strategy. For every subsequent round, the roles are reversed, the previous leader becomes the follower and the previous follower becomes the leader. The follower always knows the leader's strategy but the opposite is not true. The leader will always repeat its previous advertising strategy and both will start with the same market share they finished in the last round. The game ends if equilibrium is reached, the optimal control problem becomes infeasible or a number of predefined rounds is exceeded, whichever comes first. Figure 3.4 shows the block diagram of the game just described.

The above sequential game is equivalent to solving one of three different, yet similar, discrete optimization problems at specific rounds. The game begins with Firm A, in a monopolistic market, maximizing its net gain by solving the problem described by equations (3.3a - 3.3d). Then, whenever taking the role of follower, Firm B maximizes its profit by adopting the optimal solution of the duopoly problem given by equations (3.6a - 3.6j). Analogously, when roles are reversed and Firm A becomes the follower, it adopts the optimal advertising strategy given by the solution of the duopoly problem described by equations (3.7a - 3.7j).

Discretization of the optimal control problem for duopoly when Firm B takes

the role of follower can be written as:

$$\text{maximize}_{u_b^j} \quad J_b(x_b^j, u_b^j, N) = \sum_{k=1}^{N}(c_b x_b^j(k) - u_b^j(k)) \tag{3.6a}$$

$$\text{subject to} \quad x_a^j(k+1) = x_a^j(k) + f_{a,k}(x_a^j, x_a^j, u_a^j); \quad k = 1, ..., N, \tag{3.6b}$$

$$x_b^j(k+1) = x_b^j(k) + f_{b,k}(x_b^j, x_b^j, u_b^j); \quad k = 1, ..., N, \tag{3.6c}$$

$$x_a^j(1) = x_a^{j-1}(N+1), \tag{3.6d}$$

$$x_b^j(1) = x_b^{j-1}(N+1), \tag{3.6e}$$

$$u_a^j(k) = u_a^{j-1}(k); \quad k = 1, ..., N, \tag{3.6f}$$

$$x_a^j(k) + x_b^j(k) \leq 1 \quad k = 1, ..., N+1, \tag{3.6g}$$

$$0 \leq x_a^j(k) \leq 1; \quad k = 1, ..., N+1, \tag{3.6h}$$

$$0 \leq x_b^j(k) \leq 1; \quad k = 1, ..., N+1, \tag{3.6i}$$

$$0 \leq u_b^j(k) \leq u_{b_{max}}; \quad k = 1, ..., N \tag{3.6j}$$

Constraints (3.6d) and (3.6e), guarantee both initial market shares to be equal to the final market shares in the last round (when firm B is first introduced, $x_b^1(N+1) = 0$). Meanwhile, constraint (3.6f) forces Firm A to repeat its previous advertising strategy.

Similarly, the discretized optimization problem for duopoly when Firm A becomes the follower is given by:

$$\text{maximize}_{u_a^j} \quad J_a(x_a^j, u_a^j, N) = \sum_{k=1}^{N}(c_a x_a^j(k) - u_a^j(k)) \tag{3.7a}$$

$$\text{subject to} \quad x_a^j(k+1) = x_a^j(k) + f_{a,k}(x_a^j, x_a^j, u_a^j); \quad k = 1, ..., N, \tag{3.7b}$$

$$x_b^j(k+1) = x_b^j(k) + f_{b,k}(x_b^j, x_b^j, u_b^j); \quad k = 1, ..., N, \tag{3.7c}$$

$$x_a^j(1) = x_a^{j-1}(N+1), \tag{3.7d}$$

$$x_b^j(1) = x_b^{j-1}(N+1), \tag{3.7e}$$

$$u_b^j(k) = u_b^{j-1}(k); \quad k = 1, ..., N, \tag{3.7f}$$

$$x_a^j(k) + x_b^j(k) \leq 1 \quad k = 1, ..., N+1, \tag{3.7g}$$

$$0 \leq x_a^j(k) \leq 1; \quad k = 1, ..., N+1, \tag{3.7h}$$

$$0 \leq x_b^j(k) \leq 1; \quad k = 1, ..., N+1, \tag{3.7i}$$

$$0 \leq u_a^j(k) \leq u_{b_{max}}; \quad k = 1, ..., N \tag{3.7j}$$

The only differences aside from the objective, that now maximizes the net gain of Firm A, are constraints (3.7f) and (3.7j), which mirror equations (3.6f) and (3.6j).

Their respective NLPs are given by:

$$\underset{z}{\text{maximize}} \quad b^T z$$
$$\text{subject to} \quad r(z) = \bar{0},$$
$$Az \leq \mathcal{U}_b,$$
$$Az \geq \mathcal{L}_b$$

where

$$
b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_b \\ \vdots \\ c_b \\ 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \end{bmatrix}
\qquad
z = \begin{bmatrix} x_a^j(1) \\ \vdots \\ x_a^j(N) \\ x_a^j(N+1) \\ x_b^j(1) \\ \vdots \\ x_b^j(N) \\ x_b^j(N+1) \\ u_a^j(1) \\ \vdots \\ u_a^j(N) \\ u_b^j(1) \\ \vdots \\ u_b^j(N) \end{bmatrix}
$$

$$
r(z) = \begin{bmatrix} x_a^j(2) - x_a^j(1) - f_{a,1}(x_a^j, x_b^j, u_a^j) \\ \vdots \\ x_a^j(N+1) - x_a^j(N) - f_{a,N}(x_a^j, x_b^j, u_a^j) \\ x_b^j(2) - x_b^j(1) - f_{b,1}(x_a^j, x_b^j, u_b^j) \\ \vdots \\ x_b^j(N+1) - x_b^j(N) - f_{2,N}(x_a^j, x_b^j, u_b^j) \end{bmatrix}
$$

$$
\mathcal{L}_b = \begin{bmatrix} \mathbb{0}^{N+1} \\ x_a^{j-1} \\ \mathbb{0}^N \\ x_b^{j-1} \\ \mathbb{0}^N \\ u_a^{j-1}\mathbb{1}^N \\ \mathbb{0}^N \end{bmatrix}
\quad
A = \begin{bmatrix} I_{N+1} & I_{N+1} & & & \\ I_{N+1} & & & & \\ & & I_{N+1} & & \\ & & & I_N & \\ & & & & I_N \end{bmatrix}
\quad
\mathcal{U}_b = \begin{bmatrix} \mathbb{1}^{N+1} \\ x_a^{j-1} \\ \mathbb{1}^N \\ x_b^{j-1} \\ \mathbb{1}^N \\ u_a^{j-1}\mathbb{1}^N \\ u_{b_{max}}^N \end{bmatrix}
$$

And:

$$
\begin{aligned}
\underset{z}{\text{maximize}} \quad & a^T z \\
\text{subject to} \quad & r(z) = \bar{0}, \\
& Az \leq \mathcal{U}_a, \\
& Az \geq \mathcal{L}_a
\end{aligned}
$$

where:

$$
a = \begin{bmatrix} c_a \\ \vdots \\ c_a \\ 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\qquad
z = \begin{bmatrix} x_a^j(1) \\ \vdots \\ x_a^j(N) \\ x_a^j(N+1) \\ x_b^j(1) \\ \vdots \\ x_b^j(N) \\ x_b^j(N+1) \\ u_a^j(1) \\ \vdots \\ u_a^j(N) \\ u_b^j(1) \\ \vdots \\ u_b^j(N) \end{bmatrix}
$$

$$
r(z) = \begin{bmatrix}
x_a^j(2) - x_a^j(1) - f_{a,1}(x_a^j, x_b^j, u_a^j) \\
\vdots \\
x_a^j(N+1) - x_a^j(N) - f_{a,N}(x_a^j, x_b^j, u_a^j) \\
x_b^j(2) - x_b^j(1) - f_{b,1}(x_a^j, x_b^j, u_b^j) \\
\vdots \\
x_b^j(N+1) - x_b^j(N) - f_{2,N}(x_a^j, x_b^j, u_b^j)
\end{bmatrix}
$$

$$
\mathcal{L}_a = \begin{bmatrix}
\mathbb{0}^{N+1} \\
x_a^{j-1} \\
\mathbb{0}^N \\
x_b^{j-1} \\
\mathbb{0}^N \\
\mathbb{0}^N \\
u_b^{j-1}\mathbb{1}^N
\end{bmatrix}
\quad
A = \begin{bmatrix}
I_{N+1} & I_{N+1} & & & \\
I_{N+1} & & & & \\
& & I_{N+1} & & \\
& & & I_N & \\
& & & & I_N
\end{bmatrix}
\quad
\mathcal{U}_a = \begin{bmatrix}
\mathbb{1}^{N+1} \\
x_a^{j-1} \\
\mathbb{1}^N \\
x_b^{j-1} \\
\mathbb{1}^N \\
u_{a_{max}}^N \\
u_b^{j-1}\mathbb{1}^N
\end{bmatrix}
$$

As can be seen, the differences between the two NLPs lie in the boundary values and the elements of the constant vectors.

Figure 3.4: Block diagram of the Leader-Follower iteration.

Chapter 4

Numerical results of market share optimization using mathematical software JModelica and JuMP

In the previous chapter, we detailed the mathematical methods and software used to solve the optimal control problem based on the Vidale-Wolfe model for monopoly and Deal's extended version for duopoly. In this chapter, we present and analyze the numerical results obtained for these optimal control problems. In addition, we examine the word-of-mouth and electronic word-of-moth effects on duopolies using the modified Three Populations model discussed in Section 2.4.

Simulations were conducted as follows: Numerical results for monopoly and simultaneous co-operation in duopoly, including studies on the Three Populations model, were obtained using JModelica.org. The sequential game based on Leader-Follower iteration proposed in Section 3.3 was implemented in Julia, using the package JuMP. The choice of JuMP was made because we could not find a way, when using JModelica.org, to force one firm to repeat its previous advertising strategy in the following round – despite JModelica.org also allowing an easy, natural and concise formulation of the optimal control problem and not requiring discretization. Solver IPOPT was used with both JModelica and JuMP software in all simulations.

4.1   Monopoly results

While the scope of this work is to study the optimal control of the Vidale-Wolfe-Deal advertising model in duopolies, we begin our analysis by simulating the dynamic optimization problem for monopoly, which also happens to be the opening round in our sequential game, and then comparing our results to those proved by SETHI and THOMPSON[4] using Green's Theorem.

The initial conditions and parameters adopted in the simulation, such as maximum revenue potential (c), response to the advertising effort (r), sales exponential decay (λ), maximum advertising budget (u_{max}) and initial market share (x_0), are displayed in Table 4.1.

Table 4.1: Initial Conditions and Parameters for the Monopoly.

Initial Conditions and Parameters	Value
Maximum revenue potential c	100
Response to advertising effort r	1.2
Sales decay λ	0.2
Maximum advertising budget u_{max}	10
Initial market share x_0	10%

Figure 4.1 portrays the optimal solution [4] when $x_0 \leq x^S$ and $x^S \geq x_T$. As can be seen, the optimal trajectory can be divided into three different regions: convergence, stability and decay. During convergence, the market share goes from its initial state to an equilibrium point and the optimal advertising, u^*, takes the value of Q which is equivalent to the maximum advertising budget, u_{max}. At stability, the control action becomes constant and the dynamic system enters steady-state. Lastly, the advertising is suspended and the market share starts to decay due to the effect of λ.
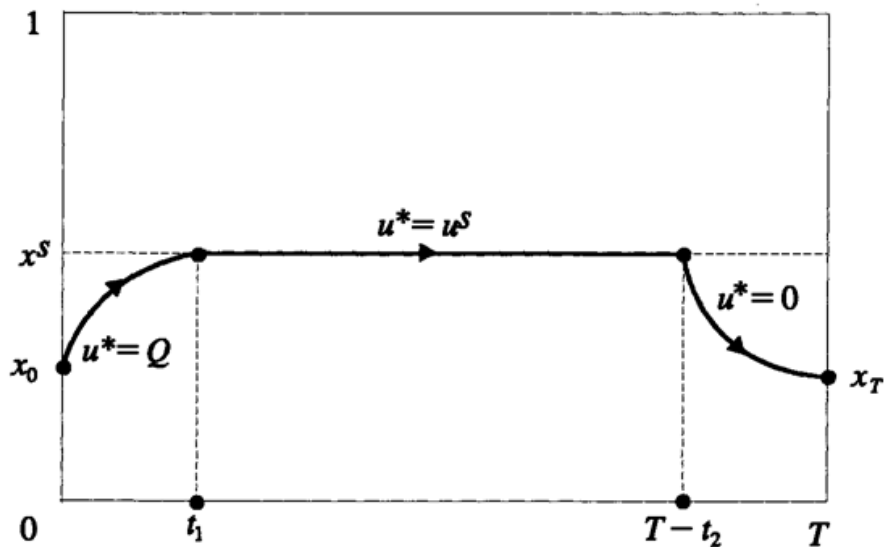
Figure 4.1: Optimal trajectory solution when $x_0 \leq x^S$ and $x^S \geq x_T$. Figure from [4], used with permission (please see Appendix C).
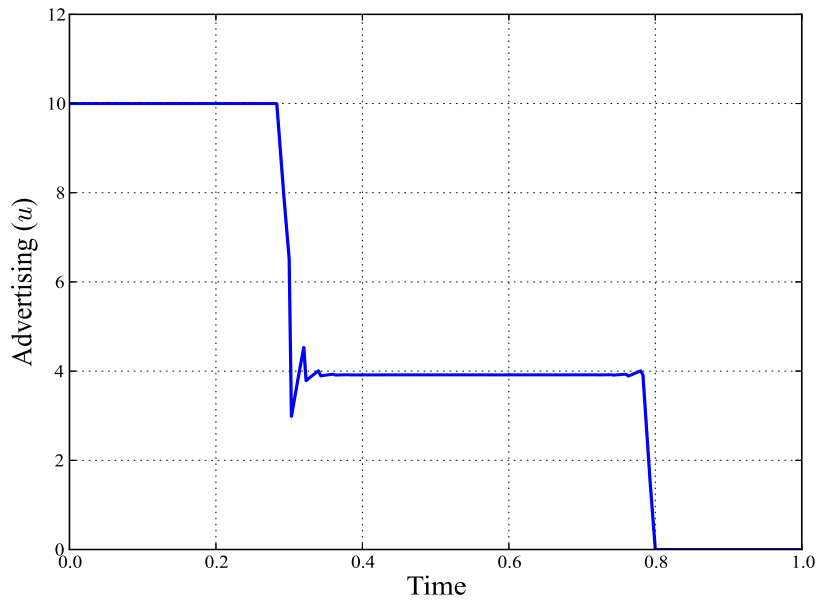
Figure 4.2: Evolution of the Advertising (control action) in a monopoly. The firm initially spends its maximum advertising limit ($u^* = u_{max} = Q = 10$) in order to reach the optimal market share as fast as possible. Then, it reduces advertising investments to $u^* = u^S = 3.9160$ in order to maintain equilibrium. In the end, it ceases investments altogether ($u^* = 0$) to increase its profit since decay of the market share is slow. This behavior matches the analytical solution portrayed in in Figure 4.1.

Figure 4.3: Evolution of the Market Share in a monopoly. The market share of the firm starts at $x_0 = 0.1$, stabilizes at $x^S = 0.9592$ (= optimal market share) and finishes at $x_T = 0.92$. This behavior matches the analytical solution portrayed in Figure 4.1.

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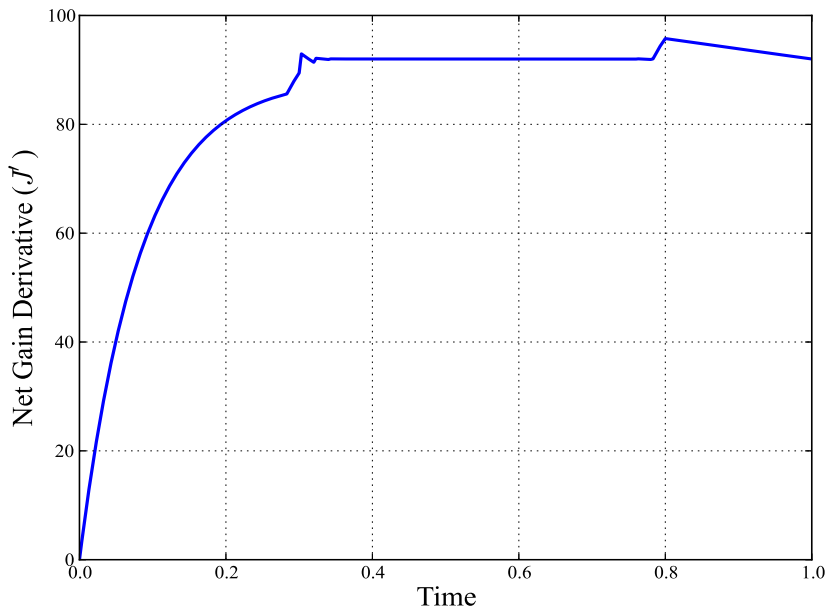
Figure 4.4: Profit curve in a monopoly. The area below the curve gives the total Net Gain (J). As can be seen, ceasing to advertise near the end of the time window ($t \approx 0.78$) increases the performance index J.

Figure 4.3 shows the evolution of the market share in response to the simulated optimal advertising strategy, depicted in Figure 4.2.

Both simulated advertising and market share behave similarly to the analytical solution in Figure 4.1. Aside from the transient oscillations when moving to a different region, the optimal advertising strategy took the value of $u_{max} = 10$ during convergence, stabilized at approximately $u^S = 3.9160$ and became zero at the end of simulation. This last behavior is explained because the slow decay of market share while ceasing advertising completely results in a momentary bigger profit than keeping the system in steady-state until the end, as shown in Figure 4.4. In response to this optimal advertising strategy, market share quickly converged from the initial value of 10% to approximately $x^S = 95.92\%$, finishing around $x_T = 92\%$ after the decay.

The equilibrium point x^S and the final market share x_T, adopting Sethi notations, depend on parameters c, r and λ. To better understand the effects of each parameter we ran a series of simulations varying one of them at a time while fixing the other two at the values presented in Table 4.1. Figures 4.5 through 4.10 plot the results of varying c, r and λ against x^S and the gap between x^S and x_T.
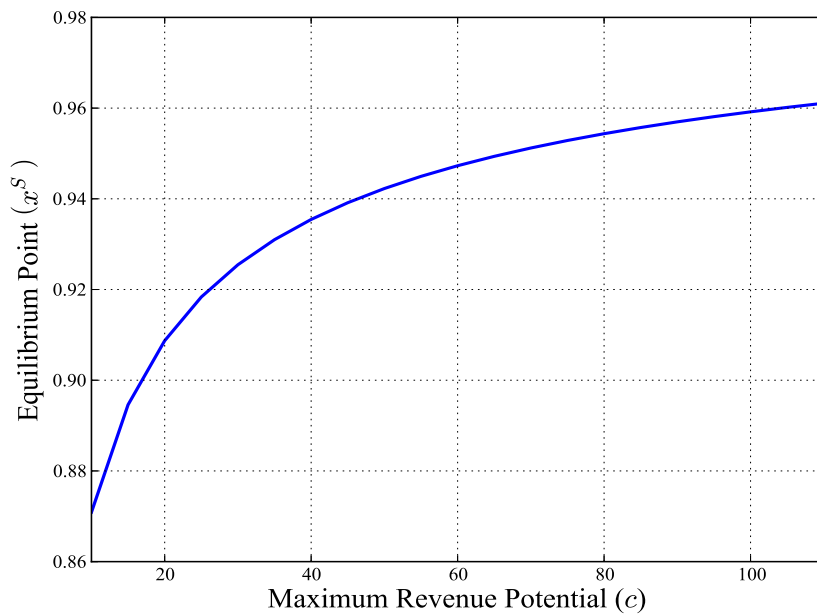
Figure 4.5: Maximum Revenue Potential (c) × Equilibrium Point (x^S). Plot shows how the equilibrium point (x^S) changes as the maximum revenue potential constant (c) varies from 10 to 110.
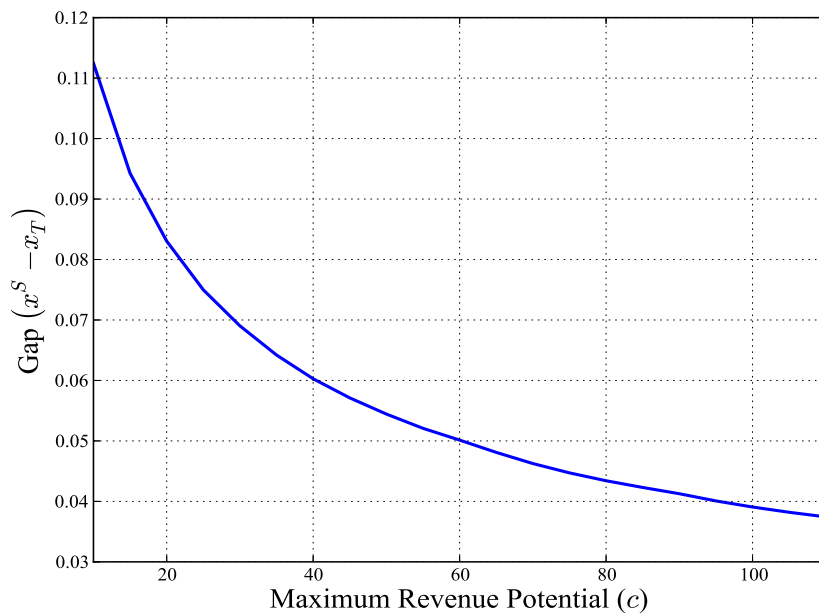
Figure 4.6: Maximum Revenue Potential (c) × Gap $(x^S - x_T)$. Plot shows how the distance (gap) between the equilibrium point (x^S) and the final market share (x_T) narrows as the maximum revenue potential constant (c) is increased from 10 to 110.
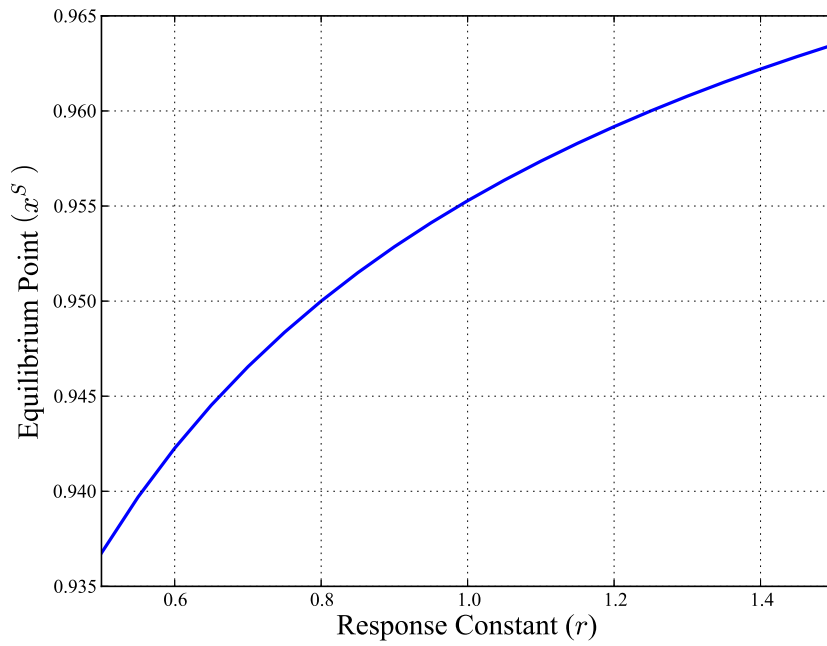
Figure 4.7: Response Constant (r) × Equilibrium Point (x^S). Plot shows how the equilibrium point (x^S) changes as the response constant (r) varies from 0.5 to 1.5.

Figure 4.8: Response Constant (c) × Gap $(x^S - x_T)$. Plot shows how the distance (gap) between the equilibrium point (x^S) and the final market share (x_T) narrows as the response constant (r) is increased from 0.5 to 1.5.
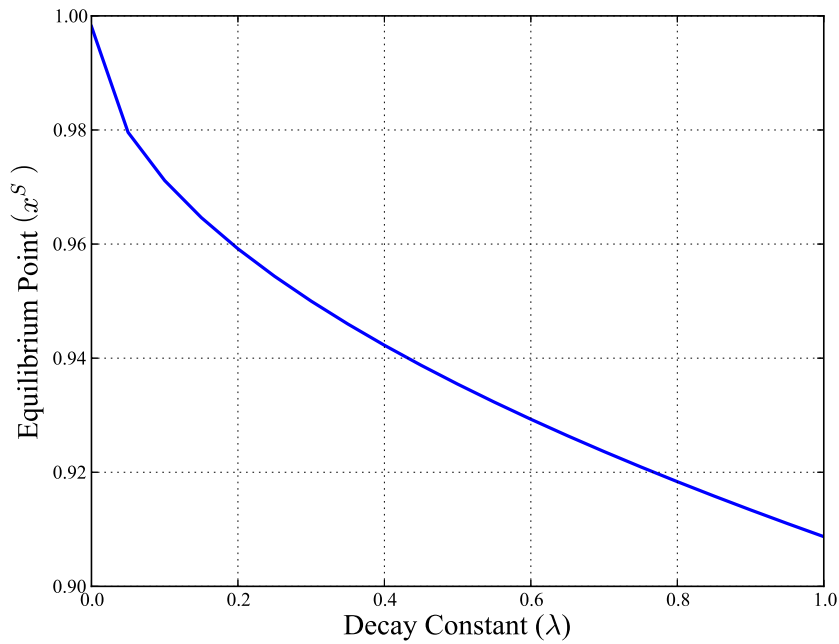
Figure 4.9: Sales Decay $(\lambda) \times$ Equilibrium Point (x^S). Plot shows how the equilibrium point (x^S) changes as the sales decay constant (λ) varies from 0 to 1.
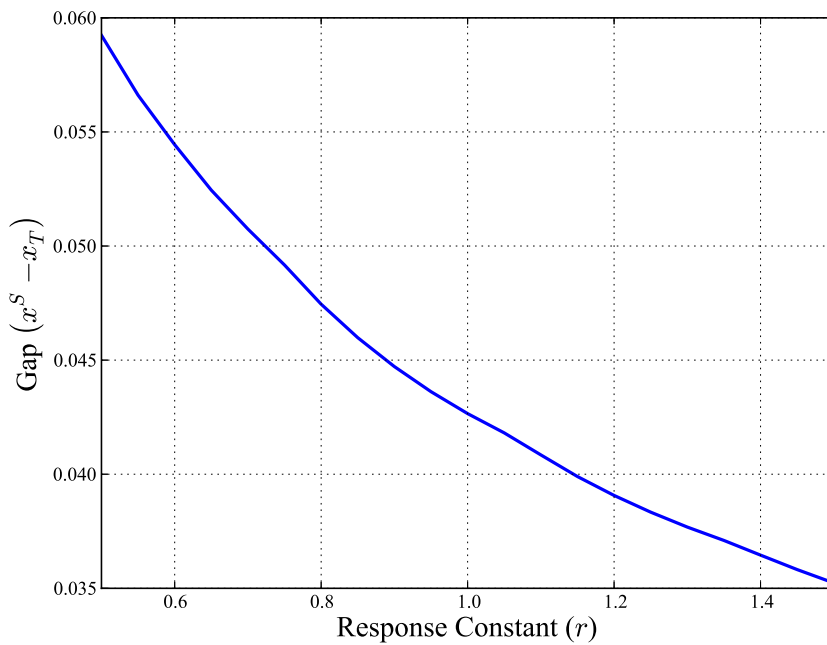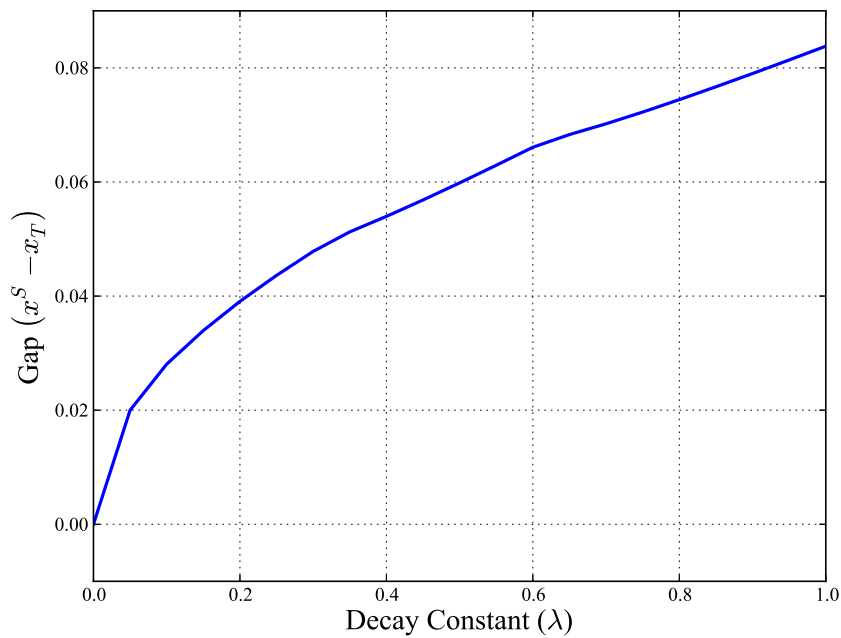

Figure 4.10: Sales Decay $(c) \times$ Gap $(x^S - x_T)$. Plot shows how the distance (gap) between the equilibrium point (x^S) and the final market share (x_T) widens as sales decay constant (λ) is increased from 0 to 1.

4.2 Duopoly results

After validating the simulated results obtained for monopoly, we extend our analysis to optimal control problems using Vidale-Volfe-Deal and Three Populations advertising models for duopoly. The analysis is split into two different parts: simultaneous co-operation (for both models) and sequential game competition (only for the Vidale-Wolfe-Deal model).

4.2.1 Simultaneous co-operation

Vidale-Wolfe-Deal model

The optimal control problem described by equations(2.12a - 2.12f) can be understood as a co-operation between two equivalent firms, labeled Firm A and Firm B, because its objective is maximizing both net profits. Co-operation between the two is said to be simultaneous because both firm's net profits are maximized at the same time. Limitation of both firms being equivalent comes from the nature of the optimization problem itself. If both profits are weighted equally and one is more profitable (better parameters) than the other, optimal solution will pursue the more favorable goal while neglecting the other entirely. This happens because the final objective is not affected by which firm the profit comes from. Thus, the problem would behave similarly to a monopoly unless constraints that regulate the market and/or a penalty for such behavior are imposed. This limitation may look strict at first, however, the manner in which companies are perceived by consumers changes within each local market. Hence, one can view two firms that share the same parameters as two firms that perform similarly on average in a more sizable market.

Table 4.2: Initial Conditions and Parameters for the Duopoly Co-op (Vidale-Wolfe-Deal).

Initial Conditions and Parameters	1	2
Firm A maximum revenue potential (c_a)	100	100
Firm B maximum revenue potential (c_b)	100	100
Cost function weight (μ)	1	1
Firm A response to advertising effort (r_a)	1.2	1.2
Firm B response to advertising effort (r_b)	1.2	1.0
Firm A sales decay (λ_a)	0.2	0.1
Firm B sales decay (λ_b)	0.2	0.2
Firm A maximum advertising budget ($u_{a_{max}}$)	10	5
Firm B maximum advertising budget ($u_{b_{max}}$)	10	20
Firm A initial market share (x_{a_0})	10%	10%
Firm B initial market share (x_{b_0})	10%	10%

Simulation of the Vidale-Wolfe-Deal duopoly model regarding co-operation were conducted adopting the parameters displayed column 1 of Table 4.2. Figures 4.11 and 4.12 show respectively the simulated market share and optimal advertising strategy for firm A when both firms are equivalent. The corresponding results for firm B can be seen in Figures 4.13 and 4.14. Both firms displayed identical results for advertising at steady-state $\left(u_a^S = u_b^S = 1.9624\right)$ and, consequently, shared the same market share at equilibrium $\left(x_a^S = x_b^S = 47.96\%\right)$ and final market share $(x_{a_T} = x_{b_T} = 46\%)$. While it is not surprising that firms A and B displayed the same behavior, it is interesting to observe that each firm employed a strategy almost equal to half the optimal advertising effort found for monopoly $(u^S = 3.9160)$, thus, achieving half the monopoly equilibrium $(x^S = 95.92\%)$ and final market share $(x_T = 92\%)$.

To verify the limitation of both firms being equivalent, we considered a scenario (column 2 of Table 4.2) were Firm A has a faster response to advertising $(r_a > r_b)$, a slower sales decay $(\lambda_a < \lambda_b)$ but a smaller maximum advertising budget $(u_{a_{max}} < u_{b_{max}})$ compared to Firm B. Figure 4.15 shows the evolution of the market shares of both firms, starting at $(x_{a_0} = x_{b_0} = 10\%)$. The market share of Firm B experiences a brief and rapid grow due to the bigger maximum budget, which allows to increase the joint profit (objective) early. Then, advertising of Firm B is ceased, its market share exponentially decreases as a direct result of the sales decay and the optimal control problem behaves similar to a monopoly of Firm A.
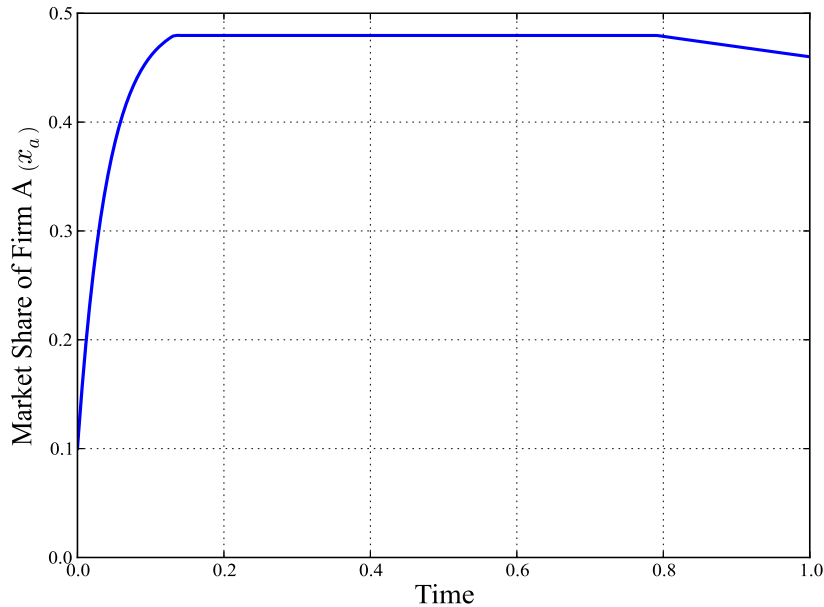
Figure 4.11: Market Share of Firm A in a duopoly: Co-operative scenario where $r_a = r_b$, $\lambda_a = \lambda_b$ and $u_{a_{max}} = u_{b_{max}}$. The values of parameters are displayed in column 1 of Table 4.2. Market share of A stars at $x_{a_0} = 0.1$ per initial condition, stabilizes at $x_a^S = 0.4796$ (= optimal market share) and finishes at $x_{a_T} = 0.46$.
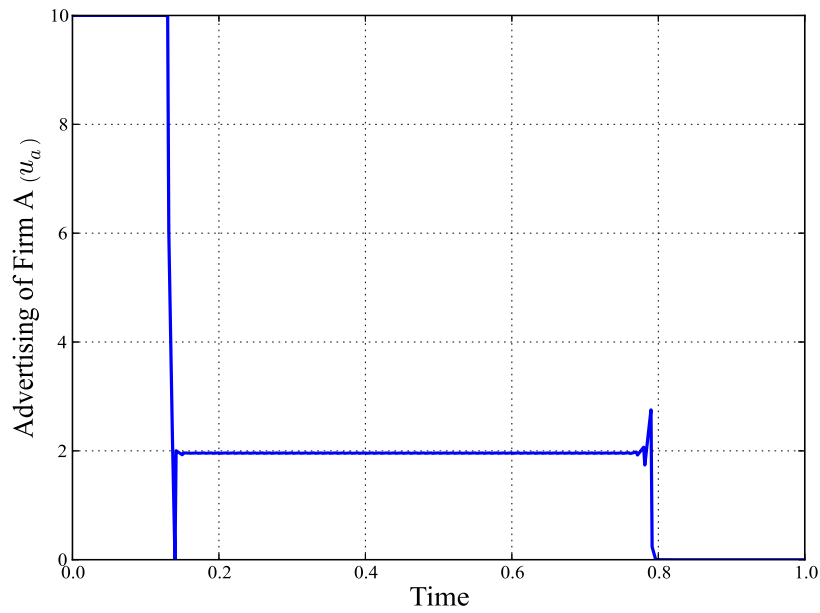
Figure 4.12: Advertising of Firm A in a duopoly: Co-operative scenario where $r_a = r_b$, $\lambda_a = \lambda_b$ and $u_{a_{max}} = u_{b_{max}}$. The values of parameters are displayed in column 1 of Table 4.2. The optimal advertising strategy begins with $u_a^* = u_{max} = 10$, is brought down to $u_a^* = u_a^S = 1.9264$ as the dynamic system enters steady-state, and is ceased ($u_a^* = 0$) near the end in order to increase its profit since decay of the market share is slow.
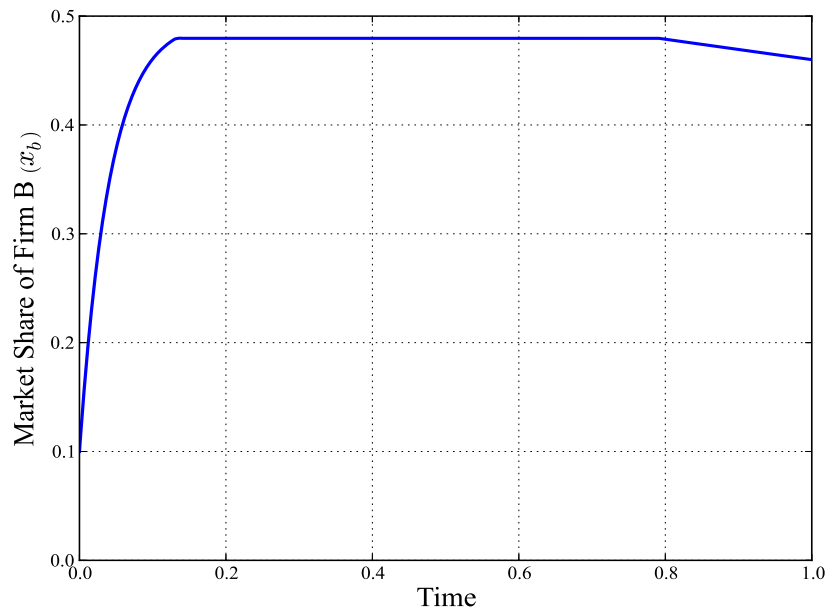
Figure 4.13: Market Share of Firm B in a duopoly: Co-operative scenario where $r_a = r_b$, $\lambda_a = \lambda_b$ and $u_{a_{max}} = u_{b_{max}}$. The values of parameters are displayed in column 1 of Table 4.2. Market share of B starts at $x_{b_0} = 0.1$ per initial condition, stabilizes at $x_b^S = 0.4796$ (= optimal market share) and finishes at $x_{b_T} = 0.46$.
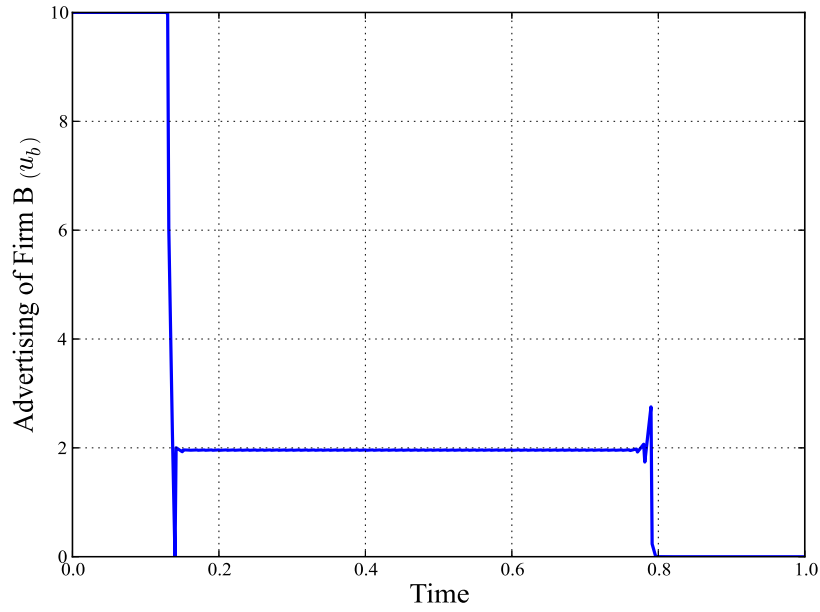
Figure 4.14: Advertising of Firm B in a duopoly: Co-operative scenario where $r_a = r_b$, $\lambda_a = \lambda_b$ and $u_{a_{max}} = u_{b_{max}}$. The values of parameters are displayed in column 1 of Table 4.2. The optimal advertising strategy begins with $u_b^* = u_{max} = 10$, is brought down to $u_b^* = u_a^S = 1.9264$ as the dynamic system enters steady-state, and is ceased ($u_b^* = 0$) near the end in order to increase its profit since decay of the market share is slow.
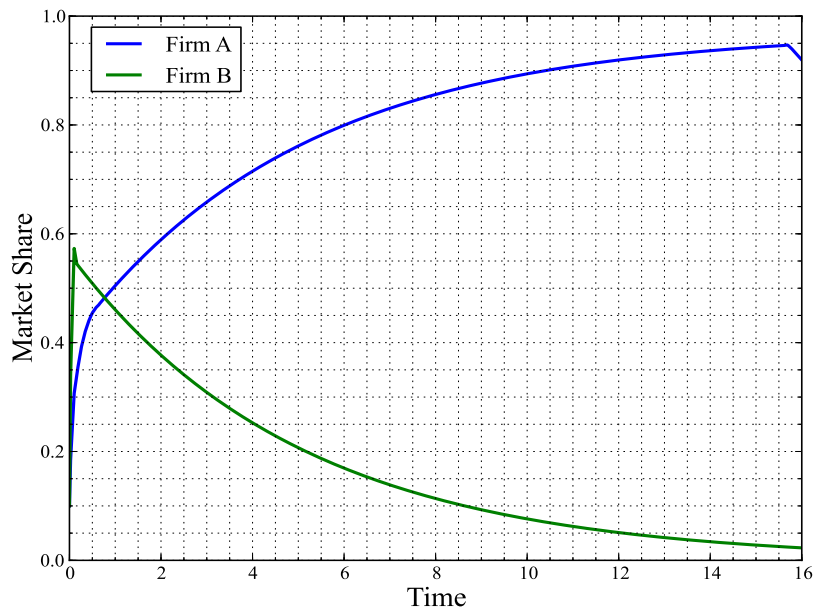

Figure 4.15: Market shares of firms A and B in a duopoly: Co-operative scenario where $r_a > r_b$, $\lambda_a < \lambda_b$ and $u_{a_{max}} < u_{b_{max}}$, resulting in crossover of market share dominance from firm B to firm A at around 2.5 time units. Parameters adopted for both firms are displayed in column 2 of Table 4.2.

Figure 4.16 shows the results of varying μ in equation (2.12a). We considered $\mu \in [0.5, 1.5]$ and used a step of 0.01 to explore the interval. It is worth noticing that even small changes in μ tilt the scales in favor of one firm or the other. For $\mu = 1$ both firms accounted for half of the total net gain (profit). For $\mu < 1$, the optimal control problem became similar to the monopoly of Firm A. Conversely, for $\mu > 1$, it became similar to the monopoly of Firm B.
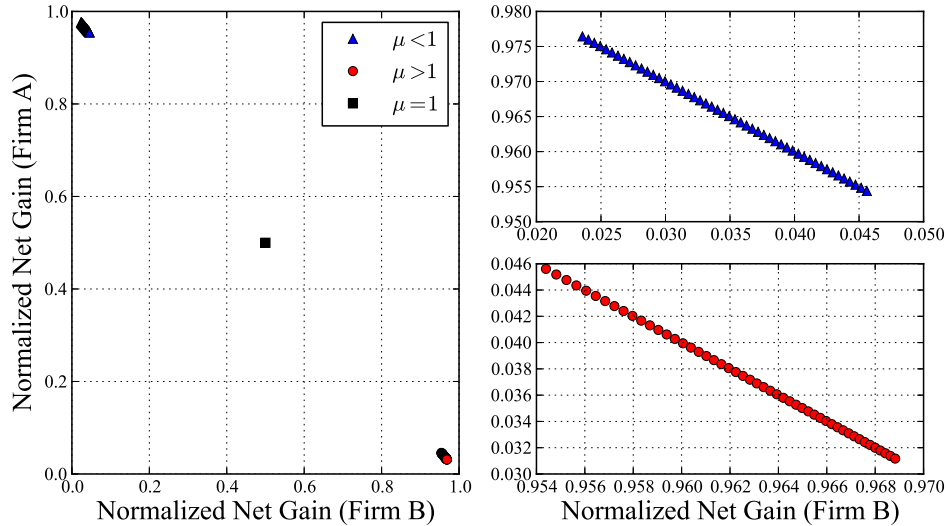
Figure 4.16: Variation of the μ parameter when $r_a = r_b$, $\lambda_a = \lambda_b$ and $u_{a_{max}} = u_{b_{max}}$. The plot on the left side shows how small variations of μ affect the normalized net gains of Firm A (ordinate) and Firm B (abscissa). For $\mu = 1$, each firm accounts for 50% of the total net gain. Plots on the right detail the changes on both net gains for $\mu < 1$ (top) and $\mu > 1$ (bottom).

Three Populations model

Results obtained for the simultaneous co-operation in duopoly using the Vidale-Wolfe-Deal model showed that the total market share of firms A and B was 95.92% $\left(x_a^S = x_b^S = 47.96\%\right)$ at equilibrium. Even when discarding the assumption of both firms being equivalent (same parameters), the sum of both market shares never surpassed 97.54%. This raises the question as to where the remaining costumers went. The Three Populations model answers the question by modeling a third population of undecided costumers and by implicitly imposing constraint (2.14), since its dynamics result in a zero-sum game (see Section 2.4).

The optimal control problem for simultaneous co-operation using the duopolistic version of the Three Populations model is described by equations(2.20a-2.20k). Repeating the methodology adopted for the Vidale-Wolfe-Deal model, we start our analysis by first investigating the optimal response to the system dynamics when both firms are considered equivalent. Column 1 of Table 4.3 comprise the initial

conditions and parameters adopted in our simulations for this scenario.

Table 4.3: Initial Conditions and Parameters for the Duopoly Co-op (Three Populations).

Initial Conditions and Parameters	1	2
Brand maximum revenue potential (c_b)	100	100
Rival maximum revenue potential (c_r)	100	100
Brand growth rate (k_b)	1.0	1.2
Rival growth rate (k_r)	1.0	1.0
Undecided-to-Brand flow rate (k_{bi})	1.0	1.0
Undecided-to-Rival flow rate (k_{ri})	1.0	1.0
Brand-Rival decay rate (k_{br})	1.0	1.0
Brand-Rival decay proportion(α)	0.5	0.5
Brand maximum advertising budget ($u_{b_{max}}$)	1	5
Rival maximum advertising budget ($u_{r_{max}}$)	1	20
Brand initial market share (x_{b_0})	10%	10%
Rival initial market share (x_{r_0})	10%	10%
Undecided initial market share (x_{i_0})	80%	80%

Figures 4.17, 4.19 and 4.21 show the market share's evolution for the brand, rival and undecided populations respectively. Albeit starting with 80% of the total market, the undecided population quickly became extinct due to the predation of both brand and rival. The two co-operating firms (brand and rival) split the market evenly, each firm finishing with 50%, which is fairly reasonable given both firms share the exact same parameters. Furthermore, the optimal advertising effort of brand and rival, shown in Figures 4.18 and 4.20, exhibited a pattern similar to the responses obtained for the Vidale-Wolfe monopoly model (Figure 4.2) and Vidale-Wolfe-Deal duopoly model (Figures 4.12 and 4.14), i.e., full advertising until reaching the optimal point of operation, followed by reducing the advertising effort just enough to stabilize the system at that optimal point (turnpike) and, then, ceasing advertising near the end of the time window in order to increase the profit without losing too much market share, due to the slow decay of the latter.

The maximum advertising budget for brand ($u_{b_{max}}$) and rival ($u_{r_{max}}$) firms were set at 1 for the sole purpose of improving the visualization of the optimal control response. Setting the values at 10, in accordance to previous simulations, only causes the optimal equilibrium, which remains the same ($x_b = x_r = 50\%$), to be reached faster.

For comparison and completeness, we also considered a scenario where brand an rival are no longer equivalent: $k_b > k_r$ and $(u_{b_{max}}) < (u_{r_{max}})$. This means brand population increases faster per unit of advertising spent meanwhile the rival firm has a bigger budget to work with. The values adopted for all parameters are shown in column 2 of Table 4.3. Figure 4.22 displays the market share's evolution for the

three populations. Again, the undecided population is preyed upon by brand and rival populations and quickly driven to extinction. In agreement with the numerical results shown in Figure 4.15, we also notice a initial surge of the rival population, which is explained by the bigger advertising budget available. As the undecided population vanishes, the dynamics shift entirely in favor of the brand (the most profitable of the two firms). Consequently, the rival population is also extinct and the whole market is comprised solely of the brand population, resulting in a monopoly.

This dissertation concludes its studies on the Three Population model for duopolies by analyzing the effects of varying the parameter α. Recapitulating Section 2.4, the parameter α determines the proportion of which brand and rival populations are depleted after each encounter between the two. At the extremes, $\alpha = 0$ indicates only the rival population is depleted and, conversely, $\alpha = 1$ indicates only the brand population is depleted. For $\alpha = 0.5$, both populations are depleted equally. Figure 4.23 shows how varying α between 0 and 1, using a step of 0.01 to explore the interval, affects the normalized net gain (profit) of each firms. Even under the premise of brand and rival being equivalent, any small deviation from $\alpha = 0.5$ tilted the scales in favor of a firm or the other.
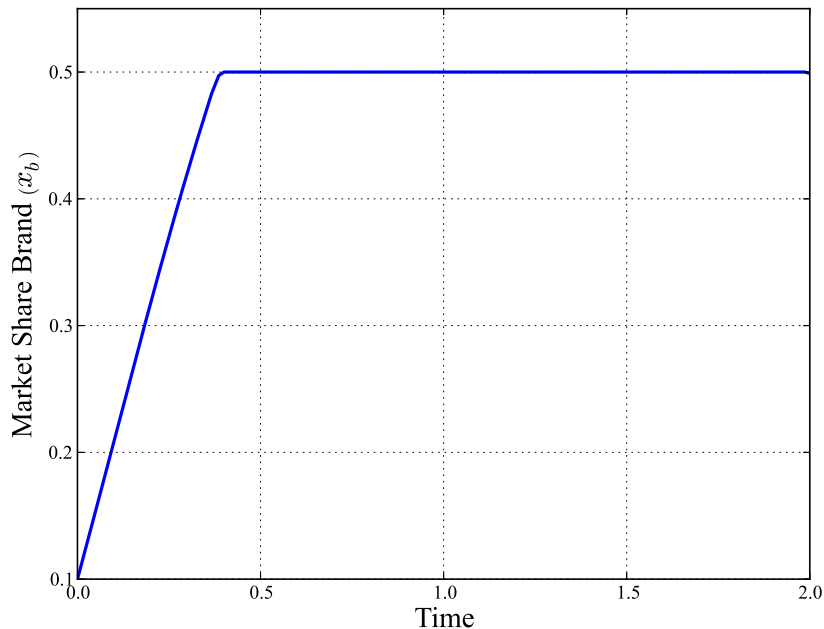
Figure 4.17: Market share of the Brand in a duopoly: Co-operative scenario where $k_b = k_r$ and $u_{b_{max}} = u_{r_{max}}$. Parameters adopted for both firms are displayed in column 1 of Table 4.3.
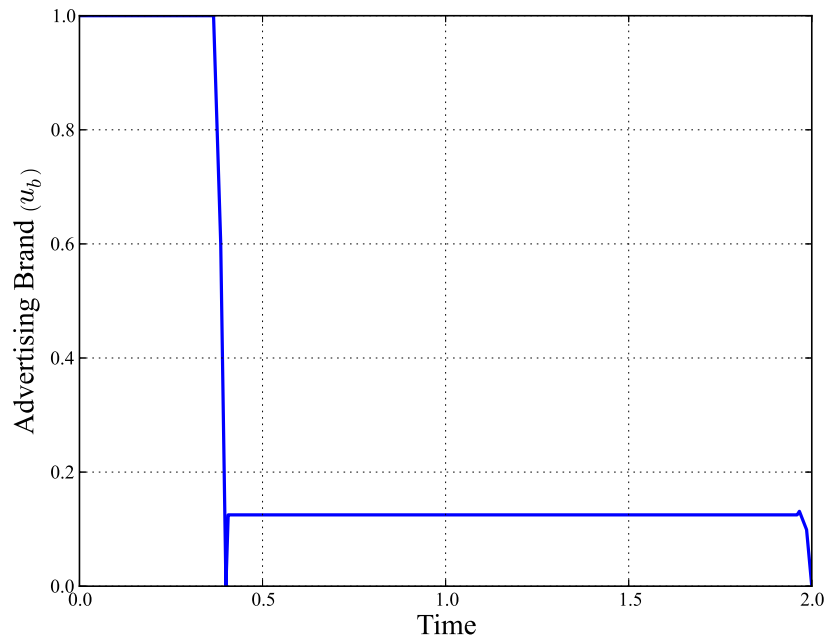
Figure 4.18: Advertising effort of the Brand in a duopoly: Co-operative scenario where $k_b = k_r$ and $u_{a_{max}} = u_{b_{max}}$. Parameters adopted for both firms are displayed in column 1 of Table 4.3. The pattern of the optimal advertising response is similar to those obtained using Vidale-Wole monopoly model and Vidale-Wolfe-Deal duopoly model.
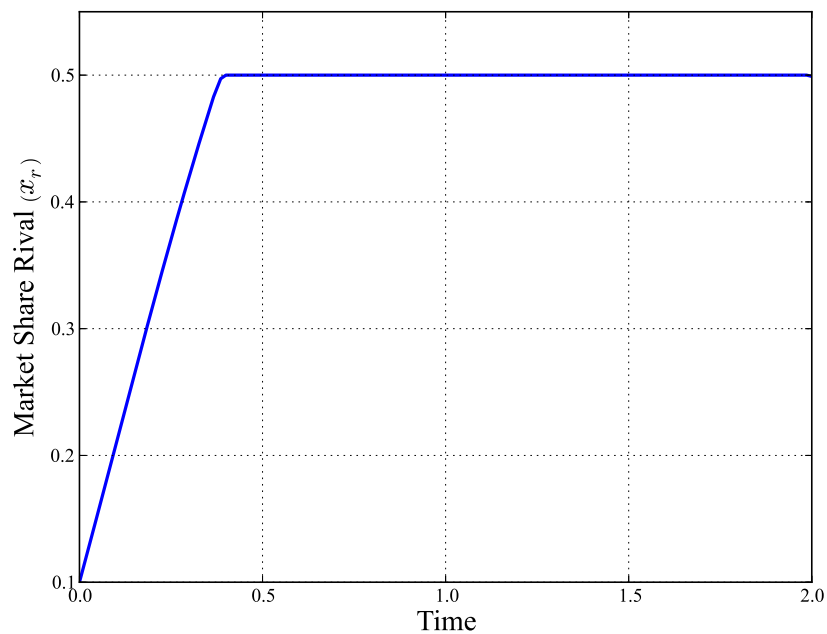
Figure 4.19: Market share of the Rival in a duopoly: Co-operative scenario where $k_b = k_r$ and $u_{b_{max}} = u_{r_{max}}$. Parameters adopted for both firms are displayed in column 1 of Table 4.3.
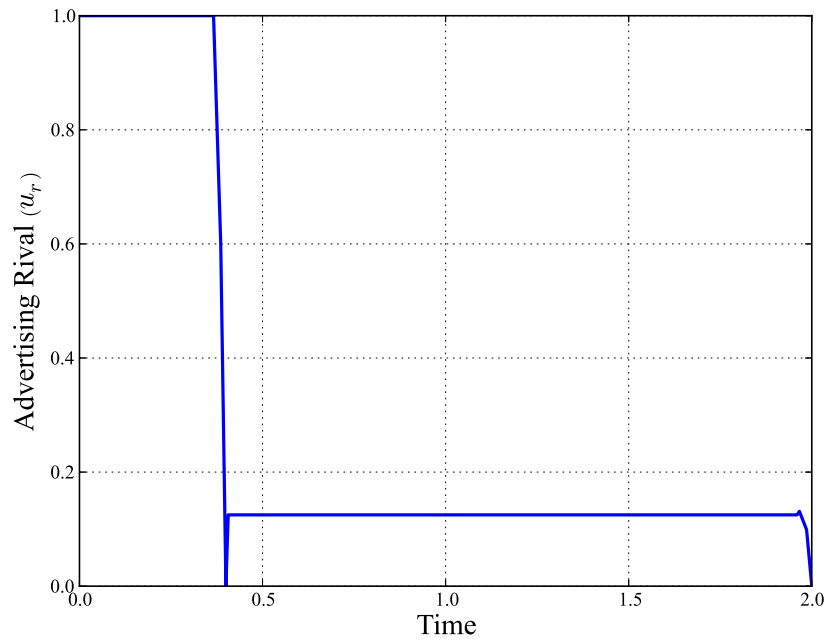
Figure 4.20: Advertising effort of the Rival in a duopoly: Co-operative scenario where $k_b = k_r$ and $u_{a_{max}} = u_{b_{max}}$. Parameters adopted for both firms are displayed in column 1 of Table 4.3. The pattern of the optimal advertising response is similar to those obtained using Vidale-Wole monopoly model and Vidale-Wolfe-Deal duopoly model.
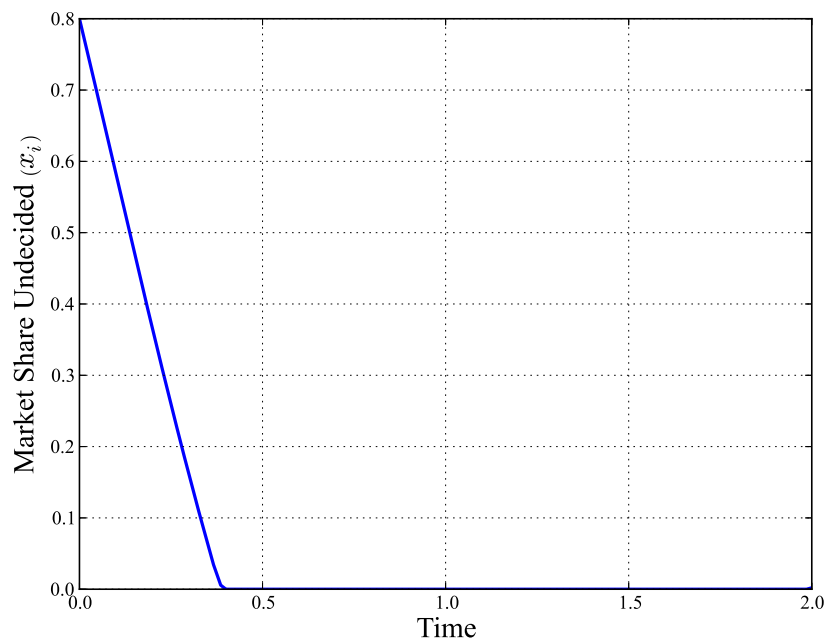
Figure 4.21: Market share of the Undecided in a duopoly: Co-operative scenario where $k_b = k_r$ and $u_{b_{max}} = u_{r_{max}}$. Parameters adopted for both firms are displayed in column 1 of Table 4.3.

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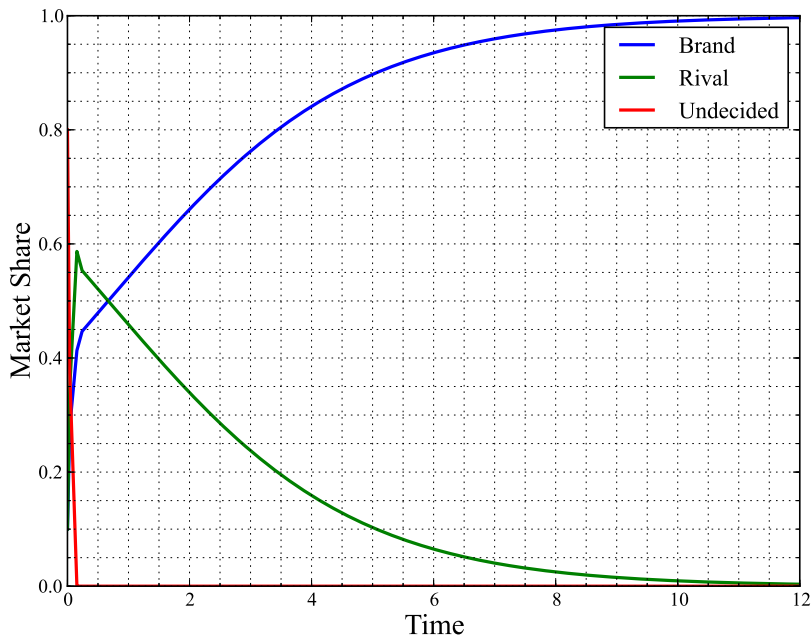
Figure 4.22: Evolution of the market shares of Brand, Rival and Undecided populations in a duopoly: Co-operative scenario where $k_b > k_r$ and $u_{b_{max}} < u_{r_{max}}$. Parameters adopted for both firms are displayed in column 2 of Table 4.3.
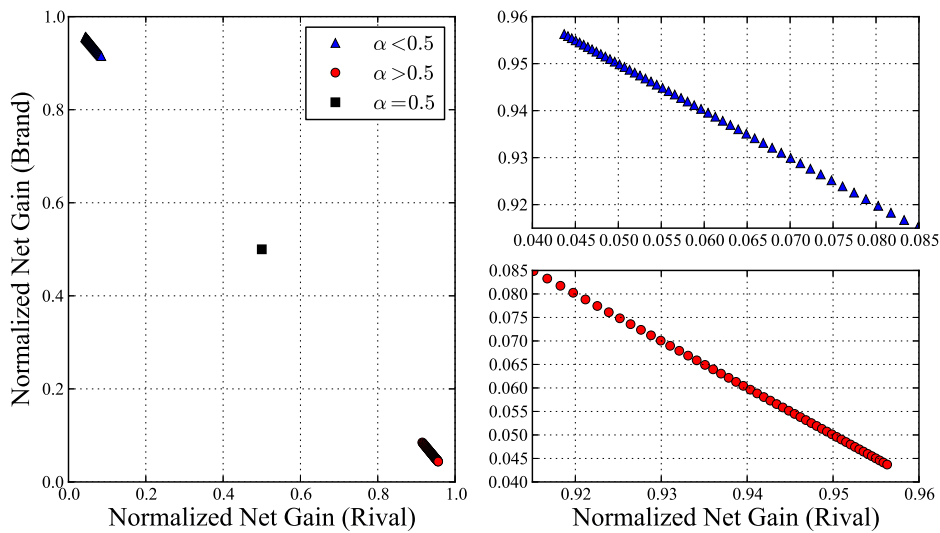
Figure 4.23: Variation of the α parameter when $k_b = k_r$ and $u_{b_{max}} = u_{r_{max}}$. The plot on the left side shows how small variations of α affect the normalized net gains of Brand (ordinate) and Rival (abscissa). For $\alpha = 0.5$, both populations are depleted equally at every encounter and each firm accounts for 50% of the total net gain. Plots on the right detail the changes on both net gains for $\alpha < 0.5$ (top) and $\alpha > 0.5$ (bottom).

4.2.2 Sequential game competition

Competition, as described in Section 3.3, was simulated considering three different scenarios. In Scenario I, Firms A and B are considered equivalent, sharing the same maximum potential revenue ($c_a = c_b$), response to advertising ($r_a = r_b$), sales decay ($\lambda_a = \lambda_b$) and maximum advertising budget ($u_{a_{max}} = u_{b_{max}}$). In Scenario II, Firm A is considered having a more recognizable brand which results in a faster response to advertising ($r_a > r_b$) and a slower sales decay ($\lambda_a < \lambda_b$). Lastly, In Scenario III, Firm A still has a more recognizable brand but Firm B has unlimited advertising budget ($u_{a_{max}} << u_{b_{max}}$). Table 4.4 summarizes the parameters and initial conditions for each scenario.

Table 4.4: Initial Conditions and Parameters for the Duopoly Competition (Stackelberg).

Initial Conditions and Parameters	Scenarios		
	I	II	III
Firm A maximum revenue potential (c_a)	100	100	100
Firm B maximum revenue potential (c_b)	100	100	100
Cost function weight (μ)	1	1	1
Firm A response to advertising effort (r_a)	1.2	1.2	1.2
Firm B response to advertising effort (r_b)	1.2	1.0	1.0
Firm A sales decay (λ_a)	0.1	0.1	0.1
Firm B sales decay (λ_b)	0.1	0.15	0.15
Firm A maximum advertising budget ($u_{a_{max}}$)	25	25	25
Firm B maximum advertising budget ($u_{b_{max}}$)	25	25	∞
Firm A initial market share at round 1 ($x_a^1(1)$)	0%	0%	0%
Firm B initial market share at round 2 ($x_b^2(1)$)	0%	0%	0%

Figure 4.24 depicts the initial market share for Firm A and Firm B in Scenario I at the beginning of each round, from rounds 1 through 150. Since their initial market share at each round is their final market share at the previous one, it is possible to observe that competition leads to an equilibrium around 50% (47.18% for Firm A and 47.13% for Firm B) market share for both firms A and B, when they adopt the strategy of maximizing their net gains. This result is not only strikingly close to the result obtained for simultaneous competition but is also supported by the results obtained by DEAL [7].

Figure 4.25 shows the average advertising effort of each firm in Scenario I at every round. We can see that the average optimal advertising of Firm A and Firm B converged at $\bar{u}_a = \bar{u}_b = 1$.

Next, we investigated what would happen when the two firms in the duopoly are not equivalent anymore. Figures 4.26 and 4.27 illustrate the results of initial market share and average advertising at every round, considering Scenario II. As one would

have expected, the equilibrium no longer is at midpoint. Firm A, which has a faster response to advertising and a slower sales decay finished with approximately 67.94% of the available market, meanwhile Firm B secured only 26.45%. The averages of both optimal advertising strategy converged, with Firm A spending 50% more than Firm B. It is worth noting that Firm B's strategy in Scenario II resembled the one seen in Scenario I, meanwhile Firm A took a more aggressive strategy, knowing it was better than the competition.

Finally, still under the premise that both firms are not in the same tier, we investigated what would change in the previous case if the lesser competitor with respect to the Vidale-Wolfe-Deal model had a much bigger advertising budget. Figure 4.28 displays the initial market share at every round for Firm A and Firm B in Scenario III. Under these circumstances, it is shown that firm B, despite having a less recognizable brand, was able to dominate the market, achieving approximately 57.43% of the final market share. Firm A, on the other hand, was strong-armed into settling for meager 36.36% of the market, even though it was considered more attractive to consumers, i.e. faster response to advertising ($r_a > r_b$) and a slower sales decay ($\lambda_a < \lambda_b$). Figure 4.29 shows the result of the average optimal advertising effort employed by both firms in Scenario III. It can be seen that in order for Firm B to overcome a better competitor, it needed to invest approximately three times the amount Firm A spent in advertising.
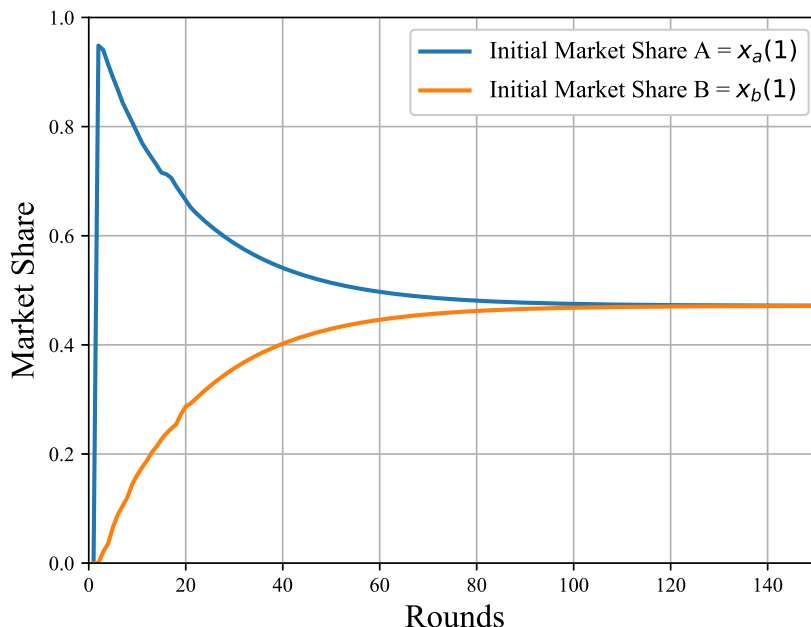
Figure 4.24: Market Share of firms A and B in a duopoly: Competitive scenario. The curves represent the evolution of market share after 150 rounds for each firm under the assumptions of Scenario I ($r_a = r_b$, $\lambda_a = \lambda_b$, $u_{a_{max}} = u_{b_{max}}$) of Table 4.4.

Figure 4.25: Advertising effort of firms A and B in a duopoly: Competitive scenario. The curves represent the average advertising made by the firms at each of the 150 rounds under the assumptions of Scenario I ($r_a = r_b$, $\lambda_a = \lambda_b$, $u_{a_{max}} = u_{b_{max}}$) of Table 4.4.
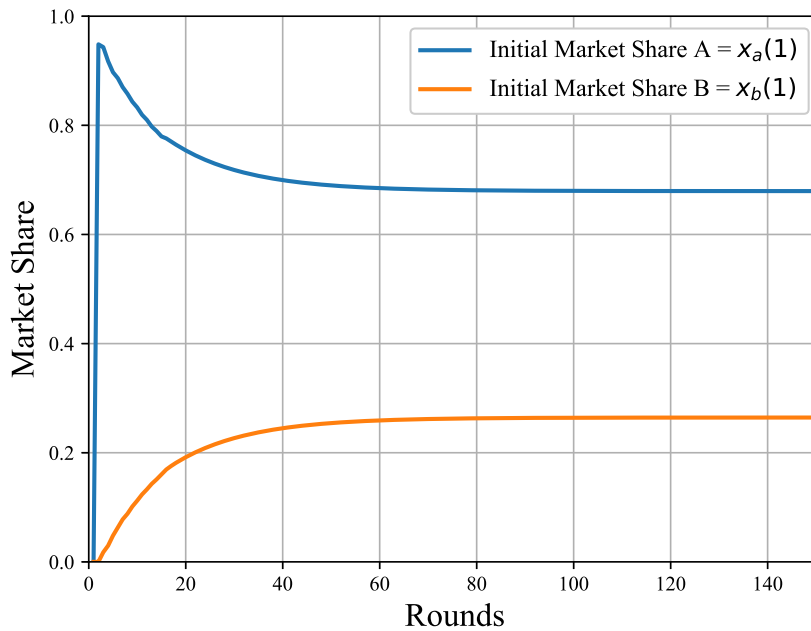
Figure 4.26: Market Share of firms A and B in a duopoly: Competitive scenario. The curves represent the evolution of market share after 150 rounds for each firm under the assumptions of Scenario II ($r_a > r_b$, $\lambda_a < \lambda_b$, $u_{a_{max}} = u_{b_{max}}$) of Table 4.4.
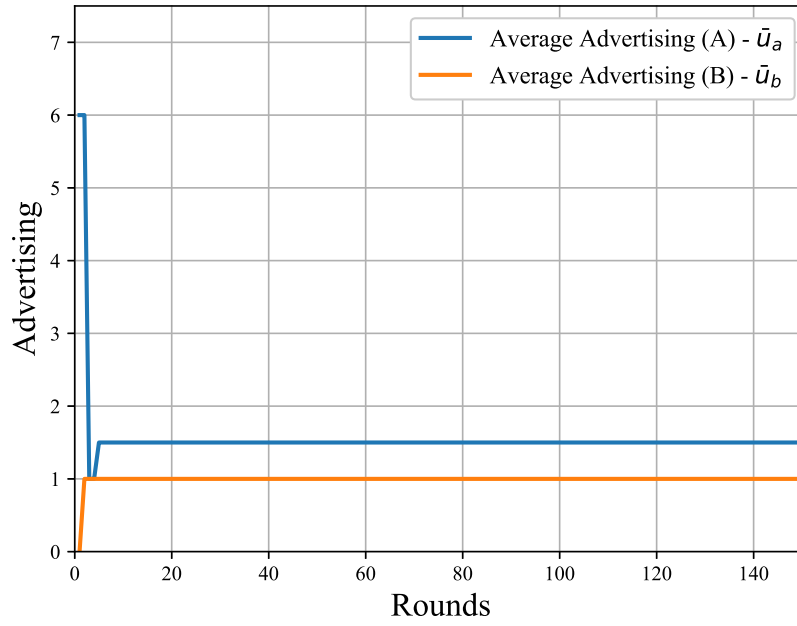
Figure 4.27: Advertising effort of firms A and B in a duopoly: Competitive scenario. The curves represent the average advertising made by the firms at each of the 150 rounds under the assumptions of Scenario II ($r_a > r_b$, $\lambda_a < \lambda_b$, $u_{a_{max}} = u_{b_{max}}$) of Table 4.4.
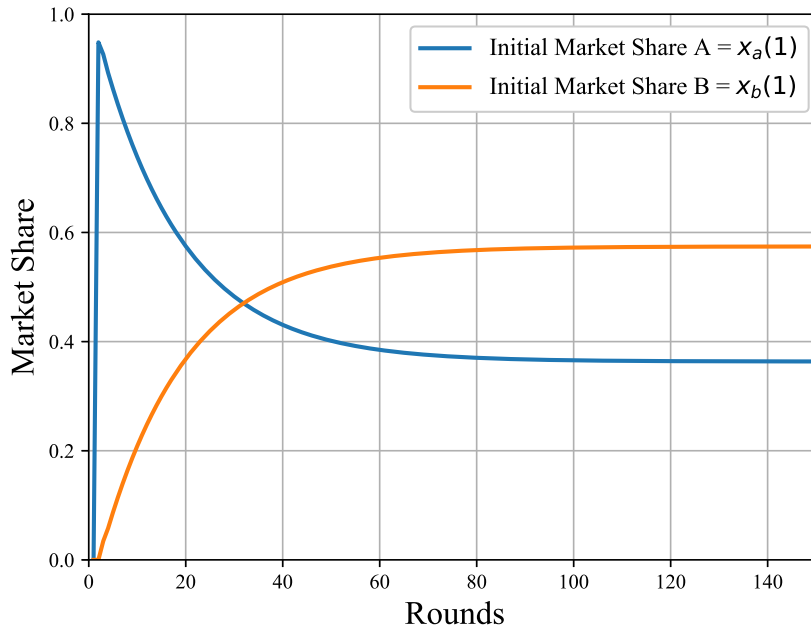
Figure 4.28: Market Share of firms A and B in a duopoly: Competitive scenario. The curves represent the evolution of market share after 150 rounds for each firm under the assumptions of Scenario III ($r_a > r_b$, $\lambda_a < \lambda_b$, $u_{a_{max}} << u_{b_{max}}$) of Table 4.4.
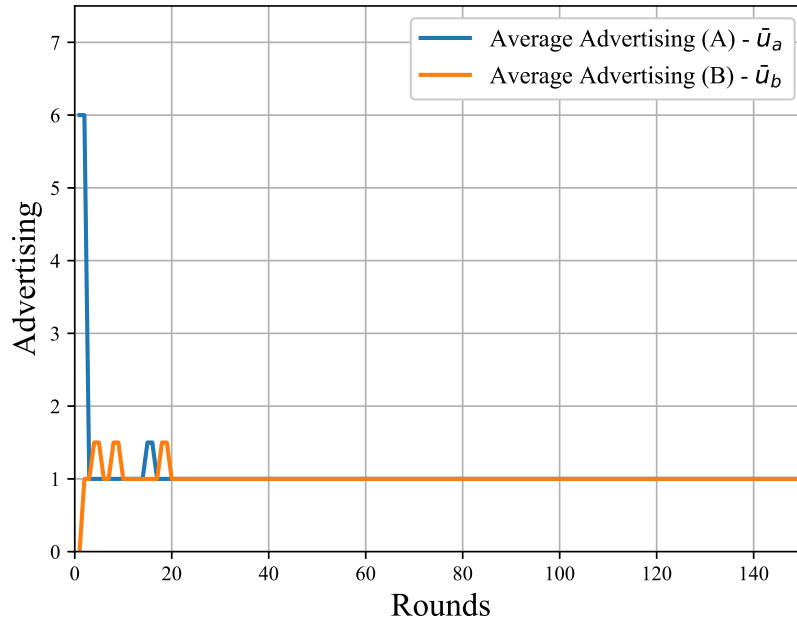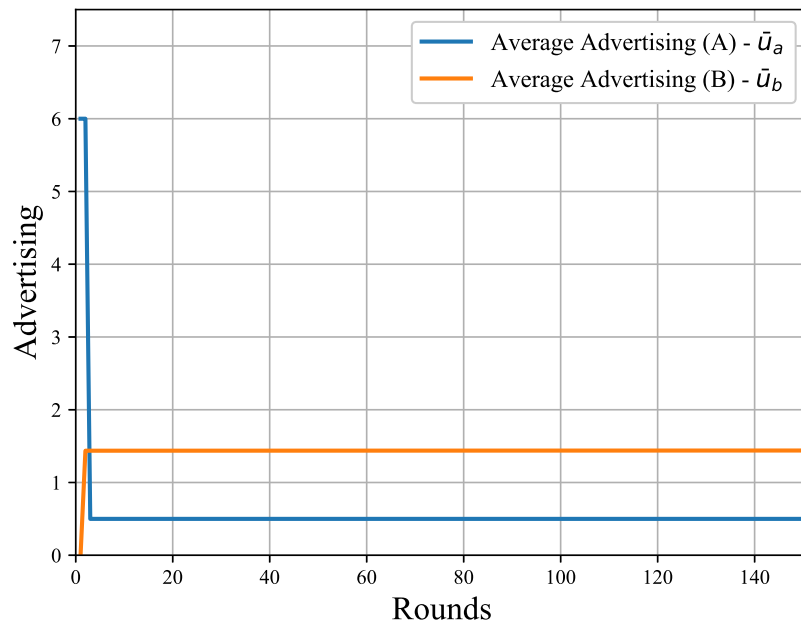
Figure 4.29: Advertising effort of firms A and B in a duopoly: Competitive scenario. The curves represent the average advertising made by the firms at each of the 150 rounds under the assumptions of Scenario III ($r_a > r_b$, $\lambda_a < \lambda_b$, $u_{a_{max}} << u_{b_{max}}$) of Table 4.4.

Chapter 5

Concluding Remarks

Numerical results obtained from the Vidale-Wolfe monopoly model 4.1 were able to verify the analytical solution found by SETHI and THOMPSON[4]. This was particularly important since the monopoly case served as cornerstone for all subsequent duopoly models, also being the starting round in the proposed sequential differential game.

The proposed duopolistic version of the Three Populations advertising model, a Lotka-Volterra type model, introduced a novel assumption that encounters between two competing brands have a negative effect on both populations and thus provoke indecision among customers, therefore increasing the undecided fraction of the market.

For both Vidale-Wolfe-Deal and Three Populations advertising models, unless the two firms are equivalent, i.e. share identical parameters, the optimal solution for the simultaneous co-operation leads to a monopoly of the firm with more favorable parameters. This result indicates that if two firms are not identical, which is usually the case, cooperation is not an optimal solution even under the adopted "cooperative" cost function because the end result is complete market domination, which might be summarized in the popular expression "business is business" (even among cooperators). When the two firms are considered equivalent, however, the numerical results for both models showed the market being evenly split between them. Furthermore, the optimal advertising effort for both models exhibited the same pattern, namely, full advertising effort until reaching the optimal point of operation, followed by reducing the advertising effort just enough to stabilize the system at that optimal point (turnpike) and, then, ceasing advertising near the end of the time window in order to increase the profit without losing too much market share, due to the slow decay of the latter. The turnpike in advertising, leading to a market share equilibrium, for the Vidale-Wolfe-Deal duopoly model is expected and is consistent with the stability analysis discussed in Section 2.2 for constant advertising effort.

In this dissertation, a sequential game based on Leader-Follower iteration was proposed and solved for the Vidale-Wolfe-Deal duopoly model. Three different scenarios were studied: Scenario I, where the two opposing firms are considered equivalent, sharing identical parameters, Scenario II, in which one of the firms has a faster response to advertising and a slower sales decay, and, lastly, Scenario III, where the firm with slower response to advertising and faster sales decay is given unlimited funds to try to compensate. Unlike the case of simultaneous co-operation, market share equilibrium was reached for all three scenarios without any firm turning the market into a monopoly. Moreover, numerical results for Scenario III showed that a bigger advertising budget under the rules established in the sequential game allows a firm to overcome its less favorable parameters. Finally, it is worth noting that simultaneous co-operation and sequential competition achieve the same market share equilibrium when both firms share identical parameters.

5.1 Contributions

The main contributions of this dissertation are as follows:

- Providing a stability analysis for the Vidale-Wolfe-Deal model showing that any control attaining final constant positive values leads to a stable equilibrium of market shares.

- Proposing a duopolistic version of the Three Populations model.

- Proposing and solving a sequential game based on Leader-Follower iteration for the Vidale-Wolfe-Deal model.

5.2 Future work

Future research directions may include:

- Studying the response, state dynamics and possible equilibria of the Three Populations advertising model for duopolies in a differential game.

- Studying the effects of modeling the Brand-Rival decay proportion, α, of the Three Populations advertising model (equation 2.16) for duopolies as a function of the type:
$$\frac{x_r}{x_b + x_r},$$
instead of assuming a constant value. Adopting such a function would cause the depletion of brand and rival populations (with the corresponding growth

of the undecided population) to be inversely proportional to their size at each encounter, which might be a more reasonable assumption.

- Considering the interactions between brand and rival populations in the duopolistic model as a Markov chain with the following possible outcomes: increase of the brand population followed by a corresponding decrease of the rival population, increase of the rival population followed by a corresponding decrease of the brand population and increase of the undecided population followed by corresponding decreases of both brand and rival populations.

- Proposing a multi-criterion cost function for the simultaneous co-operation problem in order to obtain a Pareto front.

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Appendix A

JModelica.org codes

A.1   Monopoly example

A.1.1   VW_Opt.mop

```
optimization VW_Opt (objectiveIntegrand = u - c*x, startTime = 0,
                     finalTime = 1)

// The states
        Real x(start =0.1 ,fixed=true );

// Parameters
        parameter Real c = 100;
        parameter Real lambda = 0.2;
        parameter Real r = 1.2;
        parameter Real u_max = 10;

// The control signal
        input Real u;

// System Dynamic
equation
        der(x) = r*u*(1 - x) - lambda*x;

// Box Constraints
constraint
        u >= 0;
        u <= u_max;
        x >= 0;
        x <= 1;

end VW_Opt;
```

A.2 Duopoly examples

A.2.1 VWD_Opt.mop

```
optimization VWD_Opt (objectiveIntegrand = (u1 - c*x1) +
                      mu*(u2 -c*x2), startTime = 0,
                      finalTime = 1)

// The states
        Real x1(start=0.1,fixed=true);
        Real x2(start=0.1,fixed=true);

// Parameters
        parameter Real c = 100;
        parameter Real lambda1 = 0.2;
        parameter Real lambda2 = 0.2;
        parameter Real mu = 1.0;
        parameter Real r1 = 1.2;
        parameter Real r2 = 1.2;
        parameter Real u1_max = 10;
        parameter Real u2_max = 10;

// The control signals
        input Real u1;
        input Real u2;

//System Dynamics
equation
        der(x1) = r1*u1*(1 - x1 -x2) - lambda1*x1;
        der(x2) = r2*u2*(1 - x1 -x2) - lambda2*x2;

//Box Constraints
constraint
        u1 >= 0;
        u1 <= u1_max;
        u2 >= 0;
        u2 <= u2_max;
        x1 >= 0;
        x2 >= 0;
        x1 + x2 <= 1;

end VWD_Opt;
```

A.2.2 D3pops_Opt.mop

```
optimization D3pops_Opt (objectiveIntegrand = (ub - c*xb) +
                         mu*(ur -c*xr), startTime = 0,
                         finalTime = 2)


// The states
        Real xb(start=0.10,fixed=true);
        Real xr(start=0.10,fixed=true);
        Real xi(start=0.80,fixed=true);


// Parameters
        parameter Real c = 100;
        parameter Real mu = 1;
        parameter Real alpha = 0.5;
        parameter Real ub_max = 1;
        parameter Real ur_max = 1;

        parameter Real kb = 1.2;
        parameter Real kbi = 1;
        parameter Real kbr = 1;
        parameter Real kr = 1;
        parameter Real kri = 1;


// The control signals
        input Real ub;
        input Real ur;


// System Dynamics
equation
        der(xb)= kb*ub + kbi*xb*xi - alpha*kbr*xb*xr;
        der(xr)= kr*ur + kri*xr*xi - (1-alpha)*kbr*xb*xr;
        der(xi)=-kb*ub -kr*ur - kbi*xb*xi - kri*xr*xi + kbr*xb*xr;


//Box Constraints
constraint
        ub >= 0;
        ub <= ub_max;
        ur >= 0;
        ur <= ur_max;
        xb >= 0;
        xr >= 0;
        xi >= 0;
end D3pops_Opt;
```

Appendix B

Julia code

B.1 Sequential Game

```julia
 1  using JuMP
 2  using Ipopt
 3
 4  # Parameters and initialization
 5  c = 100;
 6  lambda1 = 0.1;
 7  lambda2 = 0.1;
 8  n = 50;
 9  dt = 0.01;
10  r1 = 1.2;
11  r2 = 1.2;
12  ua_max = 25;
13  ub_max = 25;
14  xa_0 = 0.0;
15  stop = 150;
16
17  # Optimization (Monopoly)
18  VW = Model(solver = IpoptSolver())
19  @variable(VW, 0 <= x[1:n+1] <= 1)
20  @variable(VW, 0 <= u[1:n] <= ua_max)
21  @constraint(VW, x[1] == xa_0)
22  for k = 1:n
23      @constraint(VW, x[k+1] == x[k] + dt*(r1*u[k]*(1-x[k]) - lambda1*x[k]))
24  end
25
26  @objective(VW, Max, (sum(c*x[1:n] - u))*dt)
27  solve(VW)
28
29  xa = getvalue(x);
30  ua = getvalue(u);
31  marketA = xa;
32  investA = ua;
33  marketB = zeros(n + 1);
34  investB = zeros(n);
35
36  xb = marketB;
37
38  obj = getobjectivevalue(VW);
39
40  feasibleA = true;
41  feasibleB = true;
42
```

```
43  counter = 0;
44  countA = 0;
45  countB = 0;
46
47  #Leader-Follower Loop
48  while (counter < stop)&&(feasibleA)&&(feasibleB)
49
50      #Optimize Net Gain of Firm B (Duopoly)
51      VWDB = Model(solver = IpoptSolver())
52      @variable(VWDB, 0 <= x1[1:n+1] <= 1)
53      @variable(VWDB, 0 <= x2[1:n+1] <= 1)
54      @variable(VWDB, 0 <= u1[1:n] <= ua_max)
55      @variable(VWDB, 0 <= u2[1:n] <= ub_max)
56      @constraint(VWDB, x1[1] == xa[n+1])
57      @constraint(VWDB, x2[1] == xb[n+1])
58      for k = 1:n
59          @constraint(VWDB, u1[k] == ua[k])
60          @constraint(VWDB, x1[k+1] == x1[k] + dt*(r1*u1[k]*(1-x1[k]-x2[k]) -
                  lambda1*x1[k]))
61          @constraint(VWDB, x2[k+1] == x2[k] + dt*(r2*u2[k]*(1-x1[k]-x2[k]) -
                  lambda2*x2[k]))
62          @constraint(VWDB, x1[k+1] + x2[k+1] <= 1)
63      end
64      @objective(VWDB, Max, (sum(c*x2[1:n]- u2))*dt)
65      solve(VWDB)
66
67      objB = getobjectivevalue(VWDB);
68      feasibleB = !(isnan(objB));
69
70      if feasibleB
71          ub = getvalue(u2);
72          xa = getvalue(x1);
73          xb = getvalue(x2);
74
75          marketA = hcat(marketA,xa);
76          marketB = hcat(marketB,xb);
77          investA = hcat(investA,ua);
78          investB = hcat(investB,ub);
79
80          countB += 1;
81          counter += 1;
82      end
83
84      #Optimize Net Gain of Firm A (Duopoly)
85      VWDA = Model(solver = IpoptSolver())
86      @variable(VWDA, 0 <= x1[1:n+1] <= 1)
87      @variable(VWDA, 0 <= x2[1:n+1] <= 1)
88      @variable(VWDA, 0 <= u1[1:n] <= ua_max)
89      @variable(VWDA, 0 <= u2[1:n] <= ub_max)
90      @constraint(VWDA, x1[1] == xa[n+1])
91      @constraint(VWDA, x2[1] == xb[n+1])
92      for k = 1:n
93          @constraint(VWDA, u2[k] == ub[k])
94          @constraint(VWDA, x1[k+1] == x1[k] + dt*(r1*u1[k]*(1-x1[k]-x2[k]) -
                  lambda1*x1[k]))
95          @constraint(VWDA, x2[k+1] == x2[k] + dt*(r2*u2[k]*(1-x1[k]-x2[k]) -
                  lambda2*x2[k]))
96          @constraint(VWDA, x1[k+1] + x2[k+1] <= 1)
97      end
98      @objective(VWDA, Max, (sum(c*x1[1:n]- u1))*dt)
99      solve(VWDA)
100
101     objA = getobjectivevalue(VWDA);
102     feasibleA = !(isnan(objA));
103
```

```
104    if feasibleA
105        ua = getvalue(u1);
106        xa = getvalue(x1);
107        xb = getvalue(x2);
108
109        marketA = hcat(marketA,xa);
110        marketB = hcat(marketB,xb);
111        investA = hcat(investA,ua);
112        investB = hcat(investB,ub);
113
114        countA += 1;
115        counter += 1;
116    end
117 end
```

Appendix C

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C.1   Figure 3.2

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Figure C.1: Permission to use figure 2.1 of [3] located on page 39.

C.2 Figure 4.1

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Figure C.2: Permission to use figure 7.5 of Chapter 7: Applications to Marketing of [4] located on page 200.